Solutions to Assignment 2 (MA542)

Chapter 13:

6) $2 \in \mathbb{Z}$, for example.

Chapter 14:

4) Let $S \subseteq \mathbb{Z} \oplus \mathbb{Z}$ be $S = \{(a,a) : a \in \mathbb{Z}\}$.

Then (i) $(a,a)-(b,b) = (a-b, a-b) = (c,c)$ for $c = a-b$

(ii) $(a,a)(b,b) = (ab, ab) = (d,d)$ for $d = ab$

so by subring test S is a subring

But $(2,3)(a,a) = (2a, 3a) \neq (b,b)$ for any b so S is not an ideal

18) $(35) \subsetneq J \subsetneq I \subseteq \mathbb{Z}$ recall: $(m) \subseteq (n)$ iff $n \mid m$ & all ideals in $\mathbb{Z}$ are principal

so $J = (a)$, $I = (b)$ & $(35) \subsetneq J \iff a \mid 35$ & $|a| \neq 35$

$J \subsetneq I \iff b \mid a$ & $|b| \neq |a|$

$35 = 5 \cdot 7$ so J can be (5) or (7)

& I can be only $\mathbb{Z} = (1)$

29) $(x) \subseteq I$: $f \in (x) \Rightarrow f(x) = x \cdot g(x)$ $(g(x) \in \mathbb{Z}[x])$. Then $f(0) = 0\, g(0) = 0$

$I \subseteq (x)$: $f \in I$. write $f(x) = a_0 + a_1 x + \cdots + a_n x^n$. $f(0) = a_0 \Rightarrow a_0 = 0$

so $f(x) = a_1 x + \cdots + a_n x^n = x(a_1 + a_2 x + \cdots + a_n x^{n-1}) \in (x)$ QED

39) Let $f(x) = 2x+3+I$. Every $g(x) \in (\mathbb{Z}/5\mathbb{Z})[x]/I$ can be written as $g(x) = ax + b + I$

Find $g(x)$ s.t. $f(x)g(x) = 1 + I$

$$\overset{\shortparallel}{2ax^2 + (2b+3a)x + 3b + I} \qquad \text{use } x^2 + x + 2 = 0 \text{ i.e. } x^2 = -x - 2$$

$$\text{so } \overset{\shortparallel}{-2ax - 4a + (2b+3a)x + 3b + I} = (a+2b)x + 3b - 4a + I$$

so solve $\begin{cases} a+2b = 0 \\ 3b - 4a = 1 \end{cases}$ in $\mathbb{Z}/5\mathbb{Z}$ $\quad \begin{cases} a+2b = 0 \\ a+3b = 1 \end{cases} \Rightarrow$

$2b = 3b - 1$

so $b = 1$ & $a = -2b = -2 = 3$

so $\boxed{f^{-1}(x) = 3x + 1 + I}$

45) Use ideal test: Let $r,s \in \mathrm{Ann}(A)$ i.e. $ra = 0 = sa$ $\forall a \in A$

(i) $\forall a \in A$, $(r-s)a = ra - sa = 0 - 0 = 0$ so $r - s \in \mathrm{Ann}(A)$

(ii) Let $\alpha \in R$. $\forall a \in A$, $\alpha r a = \alpha(ra) = \alpha 0 = 0$ so $\alpha r \in \mathrm{Ann}(A)$

$r\alpha = \alpha r$ since R is comm. So $\mathrm{Ann}(A)$ is an ideal

46) Use ideal test: Let $r,s \in \mathrm{Nil}(A)$ i.e. $\exists m,n$ s.t. $r^m, s^n \in A$

(i) $(r-s)^{m+n} = \sum_{i=0}^{m+n} (-1)^i r^{m+n-i} s^i$. If $0 \le i \le n$, then $r^{m+n-i} = r^m r^{n-i} \in A$

If $n < i \le m+n$, then $s^i = s^{n+(i-n)} = s^n s^{i-n} \in A$

so $(r-s)^{m+n} \in A$

(ii) $(\alpha s)^m = \alpha^m s^m \in A$ $\forall \alpha \in R$

Questions:

(1) $(x+I)^2 = x^2+I = 0+I$ so $x+I$ is nilpotent.

(2) ($\Rightarrow$) let $b=1-e$, Then $eb = e(1-e) = e^2-e = e-e = 0$ & $e+b = e+1-e = 1$

($\Leftarrow$) $e = 1-b$ so $e^2 = ee = e(1-b) = e-eb = e$ QED

(3) (a) $f(g) = \begin{cases} 1 & g=e \\ 0 & \text{otherwise} \end{cases}$, let $h \in R[G]$, $h: G \to R$

$$(fh)(g) = \sum_{\substack{x,y \in G \\ xy=g}} f(x)h(y) = \sum_{y \in G} h(y) = h(g) \quad \Rightarrow fh = h \ \forall h$$

$f(x)=0$ unless $x=e$ $\quad$ $y=g$ is the only y s.t. $ey=g$

$$(hf)(g) = \sum_{\substack{x,y \in G \\ xy=g}} h(x)f(y) = \sum_{\substack{x \in G \\ xe=g}} h(x) = h(g) \quad \Rightarrow hf = h \ \forall h$$

by uniqueness of identity, this f is the identity

In other notation, it's easier: $f = 1 \cdot e$, $h = \sum_{g \in G} a_g \cdot g$ & $fh = \sum_{g \in G} a_g \cdot eg = \sum_{g \in G} a_g \cdot g = h$

& $hf = \sum_{g \in G} a_g \cdot 1 \cdot ge = \sum_{g \in G} a_g g = h$

(b) $h_1, h_2 \in I_G$ (i) $(h_1 - h_2)(g) = h_1(g) - h_2(g)$

so $\sum_{g \in G} (h_1-h_2)(g) = \sum_{g \in G} \left(h_1(g) - h_2(g) \right) = \sum_{g \in G} h_1(g) - \sum_{g \in G} h_2(g) = 0 - 0 = 0$

(ii) $h \in R[G]$, $\sum_{g \in G} (hh_1)(g) = \sum_{g \in G} \sum_{\substack{x,y \in G \\ xy=g}} h(x)h_1(y)$

$$= \sum_{g \in G} \sum_{x \in G} h(x)h_1(x^{-1}g) = \sum_{(g,x) \in G \times G} h(x)h_1(x^{-1}g)$$

as (g,x) run over $G \times G$, $(x, x^{-1}g)$ run over $G \times G$, too

$$= \sum_{(x,y) \in G \times G} h(x)h_1(y) = \sum_{x \in G} h(x) \sum_{y \in G} h_1(y) = \left(\sum_{x \in G} h(x) \right) \cdot 0 = 0$$

so $hh_1 \in I_G$

similarly $h_1 h \in I_G$

(c) Let $\varphi: R[G] \to R$ be $\varphi\left(\sum_{g \in G} a_g g \right) = \sum_{g \in G} a_g$ (so $I_G = \{ h \in R[G] \text{ s.t. } \varphi(h) = 0 \}$)

let $J = (g-1)_{g \in G, g \neq e}$

let $G = \{ e = g_0, g_1, \dots, g_n \}$ so $J = \left\{ \sum_{i=1}^{n} a_i (g_i - 1) \ : a_i \in R \right\}$

(3)(c) (cont'd):

$J \subseteq I_G$: $h \in J$, then $h = \sum_{i=1}^{n} a_i(g_i - 1) = \left(-\sum_{i=1}^{n} a_i\right) \cdot e + \sum_{i=1}^{n} a_i g_i$

so $\varphi(h) = \left(-\sum_{i=1}^{n} a_i\right) + \sum_{i=1}^{n} a_i = 0$ so $h \in I_G$

$I_G \subseteq J$: $h \in I_G$, then $h = \sum_{i=0}^{n} b_i g_i$ with $\sum_{i=0}^{n} b_i = 0$. Thus $b_0 = -\sum_{i=1}^{n} b_i$

so $h = \sum_{i=1}^{n} b_i(g_i - 1) \in J$ QED