Solutions to assignment 5 – MA542

Chapter 17:

2) If $f(x)=g(x)h(x)$ is a non-trivial factorization over D (i.e. $g,h \in D[x]$ & $g,h \notin (D[x])^\times$)
Since f is irred over F, either g or $h \in (F[x])^\times$
~~wlog $g \in (F[x])^\times$ so~~ wlog $g \in (F[x])^\times$ so $\boxed{\deg g = 0}$
Answer: one of the factors has degree 0.

10) a) use Eisenstein criterion with $p=3$. $p \nmid 1$, $p \mid 9$, $p \mid 12$, $p \mid 6$ but $p^2 \nmid 6$
$\Rightarrow$ irred. over $\mathbb{Q}$

b) Mod p irred test with $p=2$. x^4+x+1 has no roots in $\mathbb{Z}/2\mathbb{Z}$ so the only possible factors are of degree 2. The only irred. poly. over $\mathbb{Z}/2\mathbb{Z}$ of degree 2 is x^2+x+1.
$(x^2+x+1)^2 = x^4+x^2+1 \neq x^4+x+1 \Rightarrow x^4+x+1$ is irred over $\mathbb{Z}/2\mathbb{Z}$
$\Rightarrow$ " " " " $\mathbb{Q}$

c) Eisenstein criterion with $p=3$. $p \nmid 1$, $p \mid 3$, $p \mid 3$ but $p^2 \nmid 3$
$\Rightarrow$ irred over $\mathbb{Q}$

d) replace x with $x-1$ & use Eis. criterion with $p=5$: $(x-1)^5+5(x-1)^2+1 = x^5+\left(\sum_{i=1}^{4}\binom{5}{i}(-1)^{4-i}x^i\right)+5x^2-10x+5+1$
recall $5 \mid \binom{5}{i}$ $\forall 1 \le i \le 4$
The constant coeff. is 5 so $p \mid 5$ but $p^2 \nmid 5$.
so irred over $\mathbb{Q}$

e) Clear denominators & use $p=3$ with Eis. criterion:
$14 f(x) = 5\cdot 7x^5 + 3^2\cdot 7x^4 + 2\cdot 3\cdot 5\cdot 7x^3 + 2\cdot 3x^2 + 2\cdot 3\cdot 7x + 3$
so irred over $\mathbb{Q}$.

12) It's sufficient to check for roots: $f(0)=4$, $f(1)=6$, $f(2)=10$, $f(3)=5$, $f(4)=2$, $f(5)=1$
$f(6)=2$, $f(7)=5$, $f(8)=10$, $f(9)=94$, $f(10)=4$.

18) For a given n, let $I_n = \{x^n+px+p : p \text{ prime number}\} \subseteq \mathbb{Z}[x]$
By Eisenstein criterion, every element of I_n is irred / $\mathbb{Q}$
Since there are infinitely many prime numbers, I_n is infinite. QED
(of course, many other options are possible, such as $I_n = \{x^n+p : p \text{ prime}\}$ or $I_n = \{px^n+2 : p \text{ odd prime}\}$)

Chapter 17:

30) This polynomial $= \Phi_p(-x)$ where $\Phi_p(x)$ is the p^{th} cyclotomic poly. (which is irred/$\mathbb{Q}$)

If $\exists\, g, h \in \mathbb{Q}[x]$ non-units s.t. $\Phi_p(-x) = g(x)\,h(x)$

Then $\Phi_p(x) = g(-x)\,h(-x)$ contradicting irreducibility of Φ_p

$\Rightarrow \Leftarrow$ it is irred.

Questions:

(1) Degree 1: all of them are irred by definition: $\boxed{x,\ x+1}$

Degree 2: checking for roots is sufficient: $\boxed{x^2+x+1}$

Degree 3: " " " " " : $\boxed{x^3+x+1,\ x^3+x^2+1}$

Degree 4: first check for roots: $\boxed{x^4+x+1}$, ~~x^4+x^2+1~~ $= (x^2+x+1)^2$, $\boxed{x^4+x^3+1,\ x^4+x^3+x^2+x+1}$

Δ within these check for degree two irred. factors

Degree 5: first check for roots: ~~x^5+x+1~~, $\boxed{x^5+x^2+1,\ x^5+x^3+1}$, ~~$x^5+x^4+1$~~,

$\boxed{x^5+x^3+x^2+x+1,\ x^5+x^4+x^2+x+1,\ x^5+x^4+x^3+x+1,\ x^5+x^4+x^3+x^2+1}$

Then check for decomposition into product of degree 2 & degree 3 irreducibles

$$(x^2+x+1)(x^3+x+1) = x^5+x^3+x^2+x^4+x^2+x+x^3+x+1$$

$$(x^2+x+1)(x^3+x^2+1) = x^5+x^4+x^2+x^4+x^3+x+x^3+x^2+1$$