Solutions to assignment 7 - MA542

Chapter 19:

9) let $S=\{v_1,..,v_n\}$ span V. (Note: One must assume $V\neq 0$)

Claim: If S is linearly dependent & $n>1$ Then $\exists i\in\{1,..,n\}$ s.t. $S\setminus\{v_i\}$ spans V.

proof: S is lin. dep. so $\exists a_j\in F$ not all zero s.t. $0=\sum_{j=1}^{n} a_j v_j$.

$\exists i\in\{1,..,n\}$ s.t. $a_i\neq 0$. Then $v_i=\sum_{\substack{j=1\\ j\neq i}}^{n} a_i^{-1}a_j v_j$

we want to show that $S_1=\{v_1,v_2,..,v_{i-1},v_{i+1},..,v_n\}=S\setminus\{v_i\}$ spans V.

let $v\in V$. $\exists c_i\in F$ s.t. $v=\sum_{j=1}^{n} c_j v_j$ since S spans V

$$v=\left(\sum_{\substack{j=1\\ j\neq i}}^{n} c_j v_j\right)+c_i v_i=\left(\sum_{\substack{j=1\\ j\neq i}}^{n} c_j v_j\right)+\sum_{\substack{j=1\\ j\neq i}}^{n} c_i a_i^{-1}a_j v_j=\sum_{\substack{j=1\\ j\neq i}}^{n}(c_j+c_i a_i^{-1}a_j)v_j$$

so $v\in \operatorname{Span} S_1$ so $\operatorname{Span} S_1=V$ QED

So we may keep removing vectors from S & still span V until it is linearly independent or empty.

but it can't be empty since $\operatorname{Span}\emptyset=0\subsetneq V$. So $\exists$ a subset of S that is a basis of V QED

10) let $S=\{v_1,..,v_n\}\subseteq V$ be lin. indep ($n>1$)

Claim1: if S is lin. indep. & $\operatorname{Span} S\subsetneq V$, Then $\exists w\in V$ s.t. $S\cup\{w\}$ is lin. indep.

proof: by assumption $\operatorname{Span} S\subsetneq V$ so $\exists w\in V\setminus \operatorname{Span} S$. Suppose $\exists a_i, b\in F$ s.t.

$$bw+\sum_{i=1}^{n} a_i v_i=0.$$ If $b=0$, Then $\sum_{i=1}^{n} a_i v_i=0\Rightarrow a_i=0\ \forall i$ since S is lin. indep.

if $b\neq 0$, Then $w=\sum_{i=1}^{n} b^{-1}a_i v_i$ so $w\in\operatorname{Span} S$. contradiction.

Hence $S\cup\{w\}$ is lin. indep. QED

Claim2: If $\dim V=d<\infty$ & $S\subseteq V$ has more Than d elements, Then it is lin. dep.

proof: let $S=\{v_1,..,v_n\}$, $n>d$. let $\mathcal{B}=\{u_1,..,u_d\}\subseteq V$ be a basis of V.

as in class, we may show that $\mathcal{B}_1=\{v_1,u_2,u_3,..,u_d\}$

spans V (indeed: $\exists a_i$ s.t. $v_1=\sum_{i=1}^{d} a_i u_i$ & $v_1\neq 0\Rightarrow$ wlog $a_1\neq 0$, Then

$$u_1=a_1^{-1}v_1-\sum_{i=2}^{d} a_1^{-1}a_i u_i \text{ so } \operatorname{Span}\{v_1,u_2,u_3,..,u_d\}=\operatorname{Span}\mathcal{B}=V)$$

& keep going all The way to $\mathcal{B}_d=\{v_1,v_2,..,v_d\}$ spanning V

Then $\exists b_i\in F$ s.t. $v_{d+1}=\sum_{i=1}^{n} b_i v_i\Rightarrow S$ is lin. dep. contradiction.

So use claim #1 to keep adding vectors to S. Claim 2 implies This process has to stop.

QED

22) If V is an n-dim'l v.s over $\mathbb{Z}/p$, then $V \cong (\mathbb{Z}/p)^n$ which has $\boxed{p^n \text{ elements.}}$

24) $U \cap W$: (a) $0 \in U \& 0 \in W$ so $0 \in U \cap W$ so $U \cap W \neq \emptyset$

(i) let $v_1, v_2 \in U \cap W$, then $v_1, v_2 \in U$ so $v_1 + v_2 \in U$
& $v_1, v_2 \in W$ so $v_1 + v_2 \in W$

(ii) let $c \in F$, $v \in U \cap W$, then $v \in U$ so $cv \in U$
& $v \in W$ so $cv \in W$

so $U \cap V$ is a subspace by the subspace test. QED

$U+W$: (a) $0 = 0+0$ ($\& 0 \in U \& 0 \in W$) so $0 \in U+W$ so $U+W \neq \emptyset$

(i) let $v_1, v_2 \in U+W$, then $\exists u_1, u_2 \in U$, $w_1, w_2 \in W$ s.t. $v_i = u_i + w_i$

then $v_1 + v_2 = (u_1 + u_2) + (w_1 + w_2) \in U+W$
($u_1 + u_2 \in U$, $w_1 + w_2 \in W$)

(ii) let $v \in U+W$
& let $c \in F$. then $\exists u \in U, w \in W$ s.t. $v = u + w$

$cv = c(u+w) = cu + cw \in U+W$
($cu \in U$, $cw \in W$) QED

Chapter 20:

20) suppose $c \in E \supseteq F$. by definition, $F(c)$ is the smallest subfield of E containing F & c

let $a, b \in F$, $a \neq 0$

$F(ac+b) \subseteq F(c)$: since $a, b, c \in F(c)$, field axioms $\Rightarrow ab + c \in F(c)$
so $F(c)$ contains $ab+c$ so $F(ac+b) \subseteq F(c)$

$F(c) \subseteq F(ac+b)$: $a \neq 0$ so $\exists a^{-1} \in F \subseteq F(ac+b)$ s.t. $aa^{-1} = 1$. also $\exists -b \in F \subseteq F(ac+b)$ s.t. $b + -b = 0$

thus $((ac+b) + -b)a^{-1} = c \in F(ac+b)$. so $F(ac+b)$ contains c so $F(c) \subseteq F(ac+b)$

$\Rightarrow F(c) = F(ac+b)$ QED

Questions:

(1)(a)(a) $T(0) = 0$ (indeed $\forall u \in U$, $T(u+0) = T(u) + T(0)$ so $T(u) + -T(u) = T(u) + T(0) + -T(u)$ so $T(0) = 0$)
$T(u)$

so $0 \in \text{im}T$ so $\text{im}T \neq \emptyset$

(i) let $v_1, v_2 \in \text{im}T$. then $\exists u_1, u_2 \in U$ s.t. $T(u_i) = v_i$. then $T(u_1 + u_2) = T(u_1) + T(u_2) = v_1 + v_2$
so $v_1 + v_2 \in \text{im}T$

(ii) let $c \in F$, $v \in \text{im}T$. then $\exists u \in U$ s.t. $T(u) = v$. then $T(cu) = cT(u) = cv$ so $cv \in \text{im}T$

Subspace test $\Rightarrow \text{im}T$ subspace of V QED

(1)(b)·(a) $T(0)=0$ so $0\in\ker T$ so $\ker T\neq\emptyset$

(i) let $u_1,u_2\in\ker T$, Then $T(u_1+u_2)=T(u_1)+T(u_2)=0+0=0$ so $u_1+u_2\in\ker T$

(ii) let $c\in F$, $u\in\ker T$, Then $T(cu)=cT(u)=c0=0$ so $cu\in\ker T$ QED

(c)($\Rightarrow$) suppose $\ker T\neq 0$. Then $\exists u\neq 0$ s.t. $T(u)=0$. but $T(0)=0$

so T is not injective.

($\Leftarrow$) suppose T is not injective. Then $\exists u_1,u_2\in U$, $u_1\neq u_2$ but $T(u_1)=T(u_2)$.

Then $u_1-u_2\neq 0$ & $T(u_1-u_2)=T(u_1)-T(u_2)=0$

so $u_1-u_2\in\ker T$ so $\ker T\neq 0$ QED

(2)(a) let $f(x)=\sum_{n=0}^{d} a_n x^n$, $g(x)=\sum_{n=0}^{e} b_n x^n \in F[x]$ (set $a_n=0$ for $n>d$, $b_n=0$ for $n>e$)

(i) $\frac{d}{dx}(f(x)+g(x))=\frac{d}{dx}\left(\sum_{n=0}^{\max(d,e)}(a_n+b_n)x^n\right)=\sum_{n=1}^{\max(d,e)} n(a_n+b_n)x^{n-1}$

$=\sum_{n=1}^{d} n a_n x^{n-1}+\sum_{n=1}^{e} n b_n x^{n-1}=\frac{d}{dx}f(x)+\frac{d}{dx}g(x)$

(ii) let $c\in F$ $\frac{d}{dx}(cf(x))=\frac{d}{dx}\left(\sum_{n=0}^{d} c a_n x^n\right)=\sum_{n=1}^{d} n c a_n x^{n-1}=c\sum_{n=1}^{d} n a_n x^{n-1}=c\frac{d}{dx}f(x)$

QED

(b) suppose $\frac{d}{dx}f(x)=\sum_{n=1}^{d} n a_n x^{n-1}=0$, Then $n a_n=0$ $\forall n=1,\dots,d$,

$n\neq 0$ so $a_n=0$ $\forall n=1,\dots,d$

so $\ker\frac{d}{dx}=\{a_0 : a_0\in F\}$ the constant polynomials

(3) By definition, 1_E of E is the 1_F of F. let $n=\operatorname{char} F$. Then $n1_E=n1_F=0$

so $\operatorname{char} E\leq n$. if $\exists m<n$ s.t. $m1_E=0$, Then $m1_F=0$ so $\operatorname{char} E=n$

If $\operatorname{char} F=0$, $\nexists n$ s.t. $n1_F=0$ so $\nexists n$ s.t. $n1_E=n1_F=0$ so $\operatorname{char} E=0$ QED.