MAT-203 : Equations of Planes in $\mathbb{R}^3$

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So in class, I ran out of time and wasn't able to cover the material on equations of planes. You can read up on this material in the book pages 50–51. But since this is on your homework (and it's rather important material), I thought I'd write up a quick something about it, too. You should (and will need to for the homework) also read pages 52 and 53 on the distance from a point to a plane.

Equations of Planes

We saw in class that two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ (with $\mathbf{a} \neq \alpha\mathbf{b}$, i.e. not parallel) define a plane in space by a parametrization $\{s\mathbf{a} + t\mathbf{b} : s, t \in \mathbb{R}\}$. This plane goes through the origin $\mathbf{0} = (0,0,0)$ (just set $s = t = 0$). If, instead, I want a plane going through a specific point $\mathbf{p}_0$, this would be the set

$$\{\mathbf{p}_0 + s\mathbf{a} + t\mathbf{b} : s, t \in \mathbb{R}\}.$$

What we would like to do is to define this plane as an equation $f(x, y, z) = d$, i.e. without the parameters s and t (parameterizations are definitely useful, but we need to know how to do both).

It turns out that I can also define a plane by giving a point $P_0 = (x_0, y_0, z_0)$ on it and a vector $\mathbf{n} = (a, b, c)$ orthogonal to it. What I do next will explain why.

What we will do is take an arbitrary point $P = (x, y, z)$ on our plane. Then the vector from P_0 to P, $\mathbf{P_0P} = \mathbf{p} - \mathbf{p}_0$, is a vector in the plane, and hence is orthogonal to $\mathbf{n}$. So

$$\mathbf{n} \cdot (\mathbf{p} - \mathbf{p}_0) = 0.$$

Writing this in terms of coordinates:

$$\begin{aligned}
\mathbf{n} \cdot (\mathbf{p} - \mathbf{p}_0) &= (a, b, c) \cdot (x, y, z) - (a, b, c) \cdot (x_0, y_0, z_0) \\
&= ax + by + cz - (ax_0 + by_0 + cz_0)
\end{aligned}$$

If I write

$$d = \mathbf{n} \cdot \mathbf{p}_0 = ax_0 + by_0 + cz_0 \in \mathbb{R}$$

and

$$f(x, y, z) = ax + by + cz$$

then I can rewrite the equation as

$$f(x, y, z) = ax + by + cz = d.$$

So I have expressed my arbitrary point P on my plane in terms of an equation, which is what I wanted.

Example Let $P_0 = (2, 4, -1)$ and $\mathbf{n} = (3, -1, 5)$. I want to find an equation for the plane perpendicular to $\mathbf{n}$ containing the point P_0. From what we saw above, the equation is

$$3x - y + 5z = (2, 4, -1) \cdot (3, -1, 5) = 6 - 4 - 5 = -3.$$

So the answer is

$$f(x, y, z) = 3x - y + 5z = -3$$

Remark If you want to know how you can get between the two descriptions of a plane (one by a point P_0 and two vectors in the plane $\mathbf{a}$, $\mathbf{b}$, and the other by a point P_0 and a vector $\mathbf{n}$ perpendicular to the plane), here's how:

- Say I start with P_0, $\mathbf{a}$ and $\mathbf{b}$, then all I have to do to get an orthogonal vector is let

$$\mathbf{n} = \mathbf{a} \times \mathbf{b}.$$

- If instead, we start with P_0 and $\mathbf{n}$, it's more work. We need to get the equation $f(x, y, z) = d$, and then use it to find two other points P_1 and P_2 on the plane (one could find these by plugging in values, for example $x = 0$, $y = 0$ and solving for z, or something like that). Once we have these two other points, we can let $\mathbf{a}$ be the vector from P_0 to P_1 and $\mathbf{b}$ be the vector from P_0 to P_2.