Midterm 1 – Math 241

Tuesday, February 13, 2017

This is a closed-book exam. No calculators allowed.

Justify your answers to obtain full credit (and partial credit, too).

You have 1 hour and 15 minutes.

This exam consists of 5 questions, plus a bonus question, on 10 pages.

The sheets are printed on both sides. Please verify that you have all pages.

Name: Solutions **ID#:**

	Score	Out of
Question 1		40
Question 2		5
Question 3		12
Question 4		13
Question 5		10
Bonus		5
Total		80

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1. Evaluate the following limits. If they do not exist, indicate if they are ∞ or $-\infty$. Justify your answers. (5 points each)

(a) $\lim_{x \to 1} \frac{x^2+2}{x^2+3}$ $= \frac{1+2}{1+3} = \boxed{\frac{3}{4}}$

(b) $\lim_{x \to -1} \frac{2x^2-2}{x^2-x-2}$ $= \frac{2-2}{1+1-2} = \frac{0}{0}$ MW!

$= \lim_{x \to -1} \frac{\cancel{(x+1)}}{\cancel{(x+1)}} \frac{(2x-2)}{(x-2)}$

$= \frac{-2-2}{-1-2} = \frac{-4}{-3} = \boxed{\frac{4}{3}}$

(c) $\lim_{x \to \infty} \frac{3x^2+5x-7}{x^2-4}$ $= \frac{\infty}{\infty}$ MW!

$= \lim_{x \to \infty} \frac{\cancel{x^2}}{\cancel{x^2}} \frac{(3+5/x-7/x^2)}{(1-4/x^2)}$

$= \frac{3+0-0}{1-0} = \boxed{3}$

(d) $\lim\limits_{x\to-\infty} \dfrac{x^3+2}{-x+2} = \dfrac{-\infty}{\infty}$ NW!

$$= \lim_{x\to-\infty} \frac{x^3(1+2/x^3)}{x(-1+2/x)}$$

$$= \lim_{x\to-\infty} x^2 \frac{(1+2/x^3)}{(-1+2/x)}$$

$$= \infty \cdot \frac{(1+0)}{(-1+0)} = \infty \cdot (-1) = \boxed{-\infty}$$

(e) $\lim\limits_{x\to\infty} \dfrac{x^{1/2}+x^{1/3}}{x+1} = \dfrac{\infty+\infty}{\infty} = \dfrac{\infty}{\infty}$ NW!

$$= \lim_{x\to\infty} \frac{x^{1/2}(1+1/x^{1/6})}{x(1+1/x)}$$

$$= \lim_{x\to\infty} \frac{1}{x^{1/2}} \frac{(1+1/x^{1/6})}{(1+1/x)}$$

$$= \frac{1}{\infty} \frac{(1+\frac{1}{\infty})}{(1+\frac{1}{\infty})} = 0 \cdot \frac{1}{1} = \boxed{0}$$

(f) $\lim\limits_{x\to 2} \dfrac{x+3}{x^2-4} = \dfrac{5}{4-4} = \dfrac{5}{0}$ but NW!

if $x>2$, $x^2>4$ so $x^2-4>0$ so $\lim\limits_{x\to 2^+} \dfrac{x+3}{x^2-4} = \dfrac{5}{0^+} = \infty$

if $x<2$ (near 2), $x^2<4$ so $x^2-4<0$, so $\lim\limits_{x\to 2^-} \dfrac{x+3}{x^3-4} = \dfrac{5}{0^-} = -\infty$ $\neq$

so limit $\boxed{\text{DNE}}$

(g) $\lim_{x\to 1^-} \frac{x^2-4}{(x-1)(x-3)}$ $= \frac{1-4}{(1-1)(1-3)} = \frac{-3}{0\cdot(-2)} = \frac{-3}{0}$ bitMW!

if $x<1$, $x-1<0$

so $\lim_{x\to 1^-} \frac{x^2-4}{(x-1)(x-3)} = \frac{-3}{0^-\cdot(-2)} = \frac{3}{2}\cdot -\infty = \boxed{-\infty}$

(h) $\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4}$ $= \frac{2-2}{4-4} = \frac{0}{0}$ MW!

$= \lim_{x\to 4} \frac{(\sqrt{x}-2)}{(x-4)} \frac{(\sqrt{x}+2)}{(\sqrt{x}+2)}$

$= \lim_{x\to 4} \frac{\cancel{(x-4)}}{\cancel{(x-4)}(\sqrt{x}+2)} = \lim_{x\to 4} \frac{1}{\sqrt{x}+2}$

$= \frac{1}{2+2}$

$= \boxed{\frac{1}{4}}$

2. Suppose $f(x)$ is a function that satisfies

$$1 - \cos(x) \leq f(x) \leq \sin(x), \quad \text{for } 0 < x \leq \pi/6.$$

Evaluate the limit $\lim_{x \to 0^+} f(x)$. (5 points)

$\lim_{x \to 0^+} 1 - \cos(x) = 1 - \cos(0) = 1 - 1 = 0$

// → equal, so by squeeze Thm

& $\lim_{x \to 0^+} \sin(x) = \sin(0) = 0$

$\lim_{x \to 0^+} f(x) = \boxed{0}$

3. Find the *horizontal* asymptotes for each of the following functions. (6 points each)

(a) $f(x) = 1 + \dfrac{1}{x}$

$$\lim_{x\to\infty} f(x) = 1 + \frac{1}{\infty} = 1 + 0 = 1$$

$$\lim_{x\to-\infty} f(x) = 1 + \frac{1}{-\infty} = 1 + 0 = 1$$

so $\boxed{y = 1 \text{ is only horiz. asymp}}$

(b) $f(x) = \dfrac{|x| + 1}{x + 2}$

if $x > 0$, $|x| = x$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x+1}{x+2} = \frac{\infty}{\infty} \quad \text{MW!}$$

$$= \lim_{x\to\infty} \frac{\cancel{x}(1 + 1/x)}{\cancel{x}(1 + 2/x)} = \frac{1+0}{1+0} = 1$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{-x+1}{x+2} = \frac{\infty}{\infty} \quad \text{MW!}$$

if $x < 0$, $|x| = -x$

$$= \lim_{x\to-\infty} \frac{\cancel{x}(-1 + 1/x)}{\cancel{x}(1 + 2/x)} = \frac{-1+0}{1+0} = -1$$

so horiz asymp at $\boxed{\begin{array}{l} y = 1 \\ \& \; y = -1 \end{array}}$

4. Find the *vertical* asymptotes for each of the following functions.

(a) $f(x) = \dfrac{x+1}{(x-2)^2}$ (6 points)

$x - 2 = 0$ when $x = 2$ (when $x_0 \neq 2$ $(x_0-2)^2 \neq 0$

so $\lim\limits_{x \to x_0} f(x) = f(x_0) \neq \pm\infty$

so no vert asymp at $x_0 \neq 2$)

$\lim\limits_{x \to 2} f(x) = \frac{3}{0}$ but MW!

if $x \neq 2$, $(x-2)^2 > 0$

so $\lim\limits_{x \to 2} \dfrac{x+1}{(x-2)^2} = \dfrac{3}{0^+} = \infty$ so $\boxed{\text{vert. asymp. at } x = 2}$

(b) $f(x) = \dfrac{x^2 - 3x + 2}{x^2 - 1}$ (7 points)

$x^2 - 1 = 0$ when $x = 1$ or $x = -1$ (when $x_0 \neq \pm 1$, $x_0^2 - 1 \neq 0$

so $\lim\limits_{x \to x_0} f(x) = f(x_0) \neq \pm\infty$

so no vert asymp at $x_0 \neq \pm 1$)

$\lim\limits_{x \to 1} f(x) = \dfrac{1-3+2}{1-1} = \dfrac{0}{0}$ MW!

$= \lim\limits_{x \to 1} \dfrac{\cancel{(x-1)}}{\cancel{(x-1)}} \cdot \dfrac{(x-2)}{(x+1)} = \dfrac{1-2}{1+1} = \dfrac{-1}{2} \neq \pm\infty$

$\lim\limits_{x \to -1^-} f(x) = \dfrac{1+3+2}{1-1} = \dfrac{6}{0}$ but MW!

if $x < -1$, $x^2 > 1$ so $x^2 - 1 > 0$

so $\lim\limits_{x \to -1^-} f(x) = \dfrac{6}{0^+} = \infty$

so vert asymp at $\boxed{x = -1}$

5. For each of the following, determine if the discontinuity at x_0 is removable, and if it is, write down a formula for the continuous extension of $f(x)$. (5 points each)

(a) $f(x) = \dfrac{x-2}{x^2-3x+2}$ at $x_0 = 2$

$$\lim_{x\to 2} f(x) = \frac{2-2}{4-6+2} = \frac{0}{0} \quad \text{Hw!}$$

$$= \lim_{x\to 2} \frac{\cancel{(x-2)}}{\cancel{(x-2)}} \cdot \frac{1}{(x-1)} = \frac{1}{2-1} = 1 \text{ exists!}$$

so removable discont!

extension: $\boxed{F(x) = \dfrac{1}{x-1}}$

or $\boxed{F(x) = \begin{cases} \dfrac{x-2}{x^2-3x+2} & x \neq 2 \\ 1 & x = 2 \end{cases}}$

(b) $f(x) = \begin{cases} 1-x^2, & x > 1 \\ x^2, & x < 1 \end{cases}$ at $x_0 = 1$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 1-x^2 = 1-1 = 0$$

$$\neq$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2 = 1$$

so limit DNE so

$\boxed{\text{\underline{not} a removable discont}}$

(a jump discont. in fact)

(Bonus) Show that the polynomial $f(x) = x^4 - 2x - 5$ has a root. (5 points)

$f(0) = -5$ <0

$f(2) = 16 - 4 - 5 = 7$ >0

Since every polynomial is continuous, by The Intermediate Value Theorem,

There's some x_0 between 0 & 2 where $f(x_0) = 0$