Midterm 2 – Math 241

Tuesday, March 20, 2018

This is a closed-book exam. No calculators allowed.

Justify your answers to obtain full credit (and partial credit, too).

You have 1 hour and 15 minutes.

This exam consists of 5 questions on 10 pages.

The sheets are printed on both sides. Please verify that you have all pages.

Name: Solutions **ID#:**

Circle your section:

Section 8
Mondays 10:30am

Section 9
Mondays 12:30pm

	Score	Out of
Question 1		34
Question 2		12
Question 3		20
Question 4		10
Question 5		9
Total		85

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1. In parts (a)–(f), evaluate the derivatives.

(a) The first *and* second derivatives of $f(x) = -x^4 + 2x + 1 - \dfrac{1}{x} + 3x^{4/5}$ (6 points)

$$f'(x) = -4x^3 + 2 + 0 - \frac{-1}{x^2} + 3 \cdot \frac{4}{5} x^{-1/5}$$

$$= \boxed{-4x^3 + 2 + \frac{1}{x^2} + \frac{12}{5x^{1/5}}}$$

$$f''(x) = -12x^2 + 0 + \frac{-2}{x^3} + \frac{-1}{5} \cdot \frac{12}{5x^{6/5}}$$

$$= \boxed{-12x^2 - \frac{2}{x^3} - \frac{12}{25x^{6/5}}}$$

(b) $f'(x)$, where $f(x) = x^2 \sin(x)$ (5 points)

$$f'(x) = x^2 \cos(x) + \sin(x) \cdot 2x$$

$$= \boxed{x^2 \cos(x) + 2x \sin(x)}$$

(c) $\frac{d}{dt}\left(\frac{3t}{t^2+1}\right) = \frac{(t^2+1)\cdot 3 - 3t\cdot 2t}{(t^2+1)^2}$ (5 points)

$$= \frac{3t^2+3-6t^2}{(t^2+1)^2}$$

$$= \boxed{\frac{3(1-t^2)}{(t^2+1)^2}}$$

(d) $f'(3)$, where $f(x) = x\sqrt{3+2x}$ (5 points)

$$f'(x) = x \cdot \frac{1}{\cancel{2}}(3+2x)^{-1/2} \cdot \cancel{2} + \sqrt{3+2x}$$

$$= \frac{x}{\sqrt{3+2x}} + \sqrt{3+2x}$$

$$f'(3) = \frac{3}{\sqrt{9}} + \sqrt{9}$$

$$= 1+3$$

$$= \boxed{4}$$

(e) $\frac{d}{dx}\left(\cos\left(\sqrt{x^2+3x+1}\right)\right)$ (5 points)

$$= -\sin\left(\sqrt{x^2+3x+1}\right)\cdot \frac{1}{2}(x^2+3x+1)^{-1/2}\cdot(2x+3)$$

$$= \boxed{\frac{-(2x+3)\sin\left(\sqrt{x^2+3x+1}\right)}{2\cdot\sqrt{x^2+3x+1}}}$$

(f) The derivatives $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ given that $y^2+xy+1=0$. (8 points)

$$2yy' + xy' + y\cdot 1 + 0 = 0$$

so $y'(2y+x) = -y$

so $\boxed{y' = \frac{-y}{2y+x}}$

$$y'' = \frac{(2y+x)\cdot(-y') - -y\,(2y'+1)}{(2y+x)^2} = \frac{(2y+x)\left(\frac{y}{2y+x}\right) + y\left(2\,\frac{-y}{2y+x}+1\right)}{(2y+x)^2}$$

$$= \frac{2y^2+xy - \cancel{2y^2} + \cancel{2y^2} + xy}{(2y+x)^3} = \boxed{\frac{2y(x+y)}{(2y+x)^3}}$$

2. Find an equation for the tangent line to the graphs of the following at the given point.

(a) $y = \sqrt{x^3 + 1}$ at $x = 2$. (6 points)

$$y' = \frac{1}{2}(x^3+1)^{-1/2} \cdot 3x^2$$

$$y'(2) = \frac{1}{2}(9)^{-1/2} \cdot 3 \cdot 4 \qquad y(2) = \sqrt{9} = 3$$

$$= 2 \cdot \frac{3}{3} = 2$$

so $y - 3 = 2 \cdot (x-2) = 2x - 4$

so $\boxed{y = 2x - 1}$

(b) $x^2 + xy + y^2 = 3$ at $(x, y) = (1, 1)$. (6 points)

$$2x + xy' + y \cdot 1 + 2yy' = 0$$

so $2 + y' + 1 + 2y' = 0$

so $3y' = -3$ so $y' = -1$

so $y - 1 = -1 \cdot (x - 1) = -x + 1$

so $\boxed{y = -x + 2}$

3. For parts (a)–(e), consider an object moving up and down with its position (in metres) at time t (in seconds) given by

$$s(t) = t^3 - 27t + 1, \quad t \geq 0.$$

(a) What was its speed at time $t = 1$ sec? (5 points)

$v(t) = s'(t) = 3t^2 - 27$

speed at $t=1$: $|v(1)| = |3-27| = \boxed{24 \text{ m/s}}$

(b) What was its acceleration at time $t = 1$ sec? (5 points)

$a(t) = v'(t) = 6t$

$\boxed{a(1) = 6 \text{ m/s}^2}$

(c) Did it ever change direction? If so, when? (4 points)

can change direction when $v(t)=0$

$3t^2 - 27 = 0$ so $3t^2 = 27$ so $t^2 = 9$ so $t = 3$ (since $t \geq 0$)

$v(0) = -27 < 0$ & $v(10) = 300 - 27 > 0$

so yes it changed direction at $\boxed{t = 3s}$

(d) What was its displacement between time $t = 0$ sec and time $t = 5$ sec?

(3 points)

displacement: $s(5) - s(0) = 5^3 - 27 \cdot 5 + 1 - 1$

$$= 25 \cdot 5 - 27 \cdot 5 + 0$$

$$= -2 \cdot 5$$

$$= \boxed{-10\,m}$$

(e) What was the total distance it travelled between $t = 0$ sec and $t = 5$ sec?

(3 points)

Since it changed direction at $t = 3$ sec

total dist travelled is: $|s(5) - s(3)| + |s(3) - s(0)|$

$$= |5^3 - 27 \cdot 5 + \cancel{1} - (3^3 - 27 \cdot 3 + \cancel{1})| + |3^3 - 27 \cdot 3 + \cancel{1} - \cancel{1}|$$

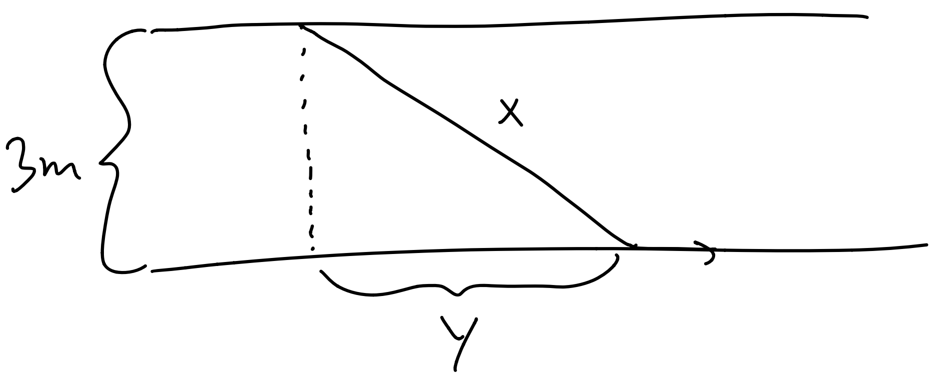
$$= |-2 \cdot 5 - (-2 \cdot 27)| + |-2 \cdot 27|$$

$$= |-10 + 54| + |-54|$$

$$= 44 + 54$$

$$= \boxed{98\,m}$$

4. You would like to figure out how fast a moving walkway in San Francisco International Airport is going. You have come equipped with a measuring tape. You measure that the ceiling is 3 metres above the surface of the walkway. You attach one end of the measuring tape to a point in the ceiling and the other end to the walkway itself. As the walkway moves, the measuring tape gets longer and you record the speed at which it's extending. When the measuring tape is 5 metres long, it is extending at a rate of 1 metre per second. How fast is the moving walkway going? (10 points)



$3^2 + y^2 = x^2$ so $0 + 2y\frac{dy}{dt} = 2x\frac{dx}{dt}$

so $\frac{dy}{dt} = \frac{x}{y}\frac{dx}{dt}$

when $x = 5\text{ m}$, $y = \sqrt{x^2 - 3^2} = \sqrt{25-9} = \sqrt{16} = 4\text{m}$

& $\frac{dx}{dt} = 1\text{m/s}$

so $\frac{dy}{dt} = \frac{5\text{m}}{4\text{m}} \cdot 1\text{m/s} = \boxed{\frac{5}{4}\text{ m/s}}$

5. The volume of a right circular cylinder of height 2 and radius r is given by $V(r) = 2\pi r^2$.

(a) Find the linearization of $V(r)$ at $r = 3$. (3 points)

$V'(r) = 4\pi r$

$V'(3) = 12\pi$ $\quad V(3) = 18\pi$

so $\boxed{L(r) = 18\pi + 12\pi(r-3)}$

(b) What is the differential of $V(r)$? (3 points)

$dV = V'(r)\,dr$

$= \boxed{4\pi r\,dr}$

(c) The radius of the cylinder starts at 3 and is stretched to 3.2. Estimate the change in volume this causes. (3 points)

$\Delta V \approx dV = 4\pi r\,dr$ with $r = 3$ & $dr = 0.2$

so $\Delta V \approx 4\pi \cdot 3 \cdot \frac{1}{5} = \boxed{\frac{12\pi}{5}}$

OR: $L(3) = 18\pi$ & $L(3.2) = 18\pi + 12\pi(3.2-3)$

so $\Delta V \approx L(3.2) - L(3) = \cancel{18\pi} + 12\pi \cdot 0.2 - \cancel{18\pi}$

$= \boxed{\frac{12\pi}{5}}$