Midterm 1 – Math 242

Friday, February 20, 2015

This is a closed-book exam. No calculators allowed.

Justify your answers to obtain full credit (and partial credit, too).

You have 50 minutes.

This exam consists of 5 questions.

The sheets are printed on both sides. Please verify that you have all pages.

Name: Solutions **ID#:**_____

	Score	Out of
Question 1		
Question 2		
Question 3		
Question 4		
Question 5		
Total		

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1. In parts (a) through (d), determine the given limit.

(a) $\lim_{x\to\infty} \frac{\ln(x)}{x} = \frac{\infty}{\infty}$ MW!

L'H: $= \lim_{x\to\infty} \frac{1/x}{1} = \frac{1}{\infty} = \boxed{0}$

(b) $\lim_{x\to\infty} \frac{e^x}{x^2+4} = \frac{\infty}{\infty}$ MW!

L'H: $= \lim_{x\to\infty} \frac{e^x}{2x} = \frac{\infty}{\infty}$ MW!

L'H: $= \lim_{x\to\infty} \frac{e^x}{2} = \frac{\infty}{2} = \boxed{\infty}$

(c) $\lim_{x\to 0^+} (1+x)^{1/x} = 1^\infty$ MW!

$\lim_{x\to 0^+} \ln\left((1+x)^{1/x}\right) = \lim_{x\to 0^+} \frac{1}{x}\ln(1+x) = \frac{0}{0}$ MW!

L'H: $= \lim_{x\to 0^+} \frac{\frac{1}{1+x}}{1} = 1$

So answer: $e^1 = \boxed{e}$

(d) $\lim_{x\to \frac{\pi}{2}^-} \left(\tan(x) - \sec(x)\right) = \infty - \infty$ MW!

Rewrite: $= \lim_{x\to \frac{\pi}{2}^-} \frac{\sin(x)}{\cos(x)} - \frac{1}{\cos(x)} = \lim_{x\to \frac{\pi}{2}^-} \frac{\sin(x)-1}{\cos(x)} = \frac{1-1}{0} = \frac{0}{0}$

L'H: $= \lim_{x\to \frac{\pi}{2}^-} \frac{\cos(x)}{-\sin(x)} = \frac{0}{-1} = \boxed{0}$

2. In parts (a) through (d), compute the given derivative.

(a) $\frac{d}{dx}((x+1)e^x) = (x+1)e^x + e^x = \boxed{e^x(x+2)}$

(b) $\frac{d}{dx}\arctan(e^x+1) = \frac{1}{1+(e^x+1)^2} \cdot e^x = \boxed{\frac{e^x}{e^{2x}+2e^x+2}}$

(c) $\frac{d}{dx}\left(\sin(x)\ln(x^2+7)\right) =$ $\sin(x)\cdot\frac{1}{x^2+7}\cdot 2x + \cos(x)\ln(x^2+7)$

(d) $\frac{d}{dx}(x+1)^{e^x}$: $y = (x+1)^{e^x}$

so $\ln(y) = \ln\left((x+1)^{e^x}\right) = e^x \ln(x+1)$

so $\frac{1}{y}\frac{dy}{dx} = e^x \cdot \frac{1}{x+1} + \ln(x+1)e^x$

so $\frac{dy}{dx} = (x+1)^{e^x} \cdot e^x \cdot \left(\frac{1}{x+1} + \ln(x+1)\right)$

3. In parts (a) through (e), evaluate the given integral.

(a) $\int \frac{\cos(x)}{1+2\sin(x)}dx = \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln|u|+C$

$u = 1+2\sin(x)$

$du = 2\cos(x)dx$

$= \boxed{\frac{1}{2}\ln|1+2\sin(x)| + C}$

(b) $\int_0^1 xe^x dx = xe^x\Big|_0^1 - \int_0^1 e^x dx = (e-0) - e^x\Big|_0^1$

$u = x \quad dv = e^x dx$

$du = dx \quad v = e^x$

$= e - (e-1)$

$= \boxed{1}$

(c) $\int_0^{\pi} x^2 \sin(x)\,dx = -x^2\cos(x)\Big|_0^{\pi} - (-2)\int_0^{\pi} x\cos(x)\,dx$

$u = x^2 \qquad dv = \sin(x)\,dx$

$du = 2x\,dx \qquad v = -\cos(x)$

$= -\pi^2\cdot(-1) - 0 + 2\int_0^{\pi} x\cos(x)\,dx$

$u = x \qquad dv = \cos(x)\,dx$

$du = dx \qquad v = \sin(x)$

$= \pi^2 + 2\left(x\sin(x)\Big|_0^{\pi} - \int_0^{\pi}\sin(x)\,dx\right)$

$= \pi^2 + 2\left((0-0) - (-\cos(x)\Big|_0^{\pi})\right)$

$= \pi^2 + 2(\cos(\pi) - \cos(0))$

$= \pi^2 + 2(-1-1) = \boxed{\pi^2 - 4}$

(d) $\int \sin^3(t)\cos^3(t)\,dt = \int \cos(t)\sin^3(t)\cos^2(t)\,dt$

$= \int \cos(t)\sin^3(t)(1-\sin^2(t))\,dt$

$u = \sin(t)$

$du = \cos(t)\,dt$

$= \int u^3(1-u^2)\,du = \int (u^3 - u^5)\,du$

$= \frac{u^4}{4} - \frac{u^6}{6} + C$

$= \boxed{\frac{\sin^4(t)}{4} - \frac{\sin^6(t)}{6} + C}$

(e) $\int \tan^3(x)dx = \int \tan^2(x)\cdot\tan(x)dx$

$= \int (\sec^2(x)-1)\tan(x)dx$

$= \int \sec^2(x)\tan(x)dx - \int \tan(x)dx$

$u = \tan(x)$
$du = \sec^2(x)dx$

$= \int u\,du - \ln|\sec(x)| + C$

$= \frac{u^2}{2} - \ln|\sec(x)| + C$

$= \boxed{\frac{\tan^2(x)}{2} - \ln|\sec(x)| + C}$

4. Find the general solution of the differential equation

$$\sqrt{t^2+1}\frac{dy}{dt} = 2te^y.$$

$$\int e^{-y}\,dy = \int \frac{2t}{\sqrt{t^2+1}}\,dt \qquad u = t^2+1$$

$$du = 2t\,dt$$

$$-e^{-y} = \int u^{-1/2}\,du = \frac{u^{1/2}}{1/2} + C_1 = 2\sqrt{t^2+1} + C_1$$

so $e^{-y} = -2\sqrt{t^2+1} + C$ (where $C = -C_1$)

so $-y = \ln(-2\sqrt{t^2+1} + C)$

so $\boxed{y = -\ln(-2\sqrt{t^2+1} + C)}$

5. Is the function $f(x) = x + e^x$ invertible? If so, what is the domain of its inverse? Justify your answer.

$f'(x) = 1 + e^x \geq 1 > 0$ for all x, so $f(x)$ is increasing for all x.

It is therefore invertible.

Domain$(f^{-1}(x))$ = Range$(f(x))$

$\lim_{x \to -\infty} x + e^x = -\infty + 0 = -\infty$

& $\lim_{x \to \infty} x + e^x = \infty + \infty = \infty$

Since $f(x)$ is continuous & starts at $-\infty$ & goes to $+\infty$, its range is all x values, i.e. $\boxed{\text{domain}(f^{-1}(x)) = (-\infty, \infty)}$

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