Midterm 2 – Math 242

Monday, April 13, 2015

This is a closed-book exam. No calculators allowed.
Justify your answers to obtain full credit (and partial credit, too).
You have 50 minutes.
This exam consists of 5 questions.
The sheets are printed on both sides. Please verify that you have all pages.

Name: Solutions **ID#:**_____

	Score	Out of
Question 1		20
Question 2		11
Question 3		13
Question 4		16
Question 5		10
Total		70

1. In parts (a) through (d), determine if the series converges or diverges. Justify.

(a) $\sum_{n=0}^{\infty} \left(\frac{-6}{5}\right)^n$

Geometric series with $r = -\frac{6}{5}$

$|r| = \frac{6}{5} > 1$

so diverges

(b) $\sum_{n=1}^{\infty} \frac{(n+1)2^n}{n!}$

$$\frac{a_{n+1}}{a_n} = \frac{(n+2)2^{n+1}}{(n+1)!} \cdot \frac{n!}{(n+1)2^n} = \frac{2(n+2)}{(n+1)(n+1)}$$

$$= \frac{2n^2\left(1+\frac{2}{n}\right)}{n^2\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)} \longrightarrow \frac{2\cdot\left(1+\frac{2}{\infty}\right)}{\infty\left(1+\frac{1}{\infty}\right)\left(1+\frac{1}{\infty}\right)} = \frac{2}{\infty} = 0$$

$0 < 1$

so converges by the ratio test

(c) $\sum_{n=2}^{\infty} \frac{2n^3+4n-7}{n^5-2}$ $\quad \frac{2n^3+4n-7}{n^5-2} \approx \frac{2n^3}{n^5} = \frac{2}{n^2}$

so compare with $\sum_{n=2}^{\infty} \frac{2}{n^2}$ $\rightarrow$ converges (p-series, $p=2>1$)

$$\frac{\frac{2n^3+4n-7}{n^5-2}}{2/n^2} = \frac{n^2 \cdot n^3 (2 + 4/n^2 - 7/n^3)}{2n^5(1-2/n^5)} \rightarrow \frac{(2+\frac{4}{\infty}-\frac{7}{\infty})}{2(1-2/\infty)} = 1$$

$0<1<\infty$ so converges by limit comparison test

(d) $\sum_{n=1}^{\infty} \frac{\sin(n^2)}{n^2}$ $\quad \left|\frac{\sin(n^2)}{n^2}\right| \leq \frac{1}{n^2}$

& $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series, $p=2>1$)

so $\sum_{n=1}^{\infty} \left|\frac{\sin(n^2)}{n^2}\right|$ converges by direct comparison test

so $\sum_{n=1}^{\infty} \frac{\sin(n^2)}{n^2}$ conv. absolutely & so converges

2. For each of the two following series, determine if it converges absolutely, conditionally, or not at all. Justify your answer.

(a) $\sum_{n=3}^{\infty} \frac{(-1)^n}{n-2}$

$\sum_{n=3}^{\infty} \frac{1}{n-2}$: $n-2 < n$ so $\frac{1}{n-2} > \frac{1}{n}$

& $\sum_{n=3}^{\infty} \frac{1}{n}$ diverges (p-series, $p = 1 \leq 1$)

so $\sum_{n=3}^{\infty} \frac{1}{n-2}$ diverges by Direct Comparison Test

$\sum_{n=3}^{\infty} \frac{(-1)^n}{n-2}$: $\lim_{n\to\infty} \frac{1}{n-2} = \frac{1}{\infty} = 0$

so converges by Alternating Series test

so the series converges conditionally

(b) $\sum_{n=1}^{\infty} \frac{5+\sin(n)}{\sqrt{n}}$

$\frac{5+\sin(n)}{\sqrt{n}} \geq \frac{4}{\sqrt{n}}$ & $\sum_{n=1}^{\infty} \frac{4}{\sqrt{n}}$ diverges

(p-series, $p = \frac{1}{2} \leq 1$)

so $\sum_{n=1}^{\infty} \frac{5+\sin(n)}{\sqrt{n}}$ diverges by Direct Comparison Test.

3. Evaluate the following two integrals. Justify as needed.

(a) $\int \frac{2x+5}{(x+1)(x+2)}dx. = I$

$$\frac{2x+5}{(x+1)(x+2)} = \frac{A}{(x+1)} + \frac{B}{(x+2)} = \frac{Ax+2A+Bx+B}{(x+1)(x+2)}$$

So $2 = A+B$ (1)

$5 = 2A+B$ (2)

(2)−(1): $3 = A$

$2 = 3+B$ so $B = -1$

So $I = \int \frac{3}{x+1}dx - \int \frac{1}{x+2}dx = \boxed{3\ln|x+1| - \ln|x+2| + C}$

(b) $\int \frac{x^2}{(1-x^2)^{3/2}} dx. =$ $\int \frac{\sin^2\theta}{(\cos^2\theta)^{3/2}} \cos\theta\, d\theta = \int \frac{\sin^2\theta}{\cos^2\theta}\, d\theta$

$= \int \tan^2\theta\, d\theta$

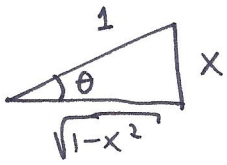
$= \int \sec^2\theta\, d\theta - \int d\theta$

$= \tan\theta - \theta + C$

$= \boxed{\frac{x}{\sqrt{1-x^2}} - \arcsin(x) + C}$

$x = \sin\theta$

$dx = \cos\theta\, d\theta$



4. For each of the three following series, determine if it converges, and if so, determine its sum. Justify your answer.

(a) $\displaystyle\sum_{n=0}^{\infty}\left(\frac{1}{5^n}+\frac{2^{n+1}}{3^n}\right) = \sum_{n=0}^{\infty}\frac{1}{5^n} + 2\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n$

$= \frac{1}{1-1/5} + 2\cdot\frac{1}{1-2/3}$

$= \frac{1}{4/5} + 2\cdot\frac{1}{1/3} = \frac{5}{4} + 6 = \boxed{\frac{29}{4}}$

(so it converges)

(b) $\displaystyle\sum_{n=0}^{\infty}\arctan(n)$ $\lim_{n\to\infty}\arctan(n) = \frac{\pi}{2} \neq 0$ so diverges by divergence test

(c) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+2}\right) = 1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4}$
$+\frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6}$
$+\frac{1}{5} - \frac{1}{7} + \frac{1}{6} - \frac{1}{8}$

telescoping

(partial sum: for $k\geq 2$ $S_k = 1 + \frac{1}{2} - \frac{1}{k+1} - \frac{1}{k+2}$ $\to 1+\frac{1}{2}$)

$= 1 + \frac{1}{2} = \boxed{\frac{3}{2}}$

(so it converges)

5. For each of the following two integrals, determine if it converges, and if so, find its value. Justify your answer.

(a) $\int_2^\infty \frac{2x}{(x^2+2)^2}dx = \int_6^\infty \frac{du}{u^2} = \frac{-1}{u}\Big|_6^\infty = \frac{-1}{\infty} - \frac{-1}{6} = \boxed{\frac{1}{6}}$

$u = x^2+2$

$du = 2x\,dx$

(So converges)

(b) $\int_2^\infty \frac{2x}{x^2+2}dx = \int_6^\infty \frac{1}{u}du = \ln|u|\Big|_6^\infty = \ln|\infty| - \ln|6|$

$= \infty$ so diverges

$u = x^2+2$

$du = 2x\,dx$

Formula sheet

Some integral formulas

- $\int \tan(x)dx = \ln|\sec(x)| + C$
- $\int \cot(x)dx = -\ln|\csc(x)| + C$
- $\int \sec(x)dx = \ln|\sec(x) + \tan(x)| + C$
- $\int \csc(x)dx = -\ln|\csc(x) + \cot(x)| + C$
- $\int \frac{1}{\sqrt{a^2 - x^2}}dx = \arcsin\left(\frac{x}{a}\right) + C$
- $\int \frac{1}{a^2 + x^2}dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$