Midterm 1 – Math 243

Friday, February 17, 2017

This is a closed-book exam. No calculators allowed.

Justify your answers to obtain full credit (and partial credit, too).

You have 50 minutes.

This exam consists of 5 questions.

The sheets are printed on both sides. Please verify that you have all pages.

Name: Solutions **ID#:** ____________

	Score	Out of
Question 1		
Question 2		
Question 3		
Question 4		
Question 5		
Total		

1. (a) For the vectors $\mathbf{u} = (4, 1, 2), \mathbf{v} = (1, -3, 1)$, compute their dot product and cross product.

$$\vec{u} \cdot \vec{v} = 4 \cdot 1 - 1 \cdot 3 + 2 \cdot 1$$
$$= 4 - 3 + 2$$
$$= \boxed{3}$$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 1 & 2 \\ 1 & -3 & 1 \end{vmatrix} = \hat{i}\begin{vmatrix} 1 & 2 \\ -3 & 1 \end{vmatrix} - \hat{j}\begin{vmatrix} 4 & 2 \\ 1 & 1 \end{vmatrix} + \hat{k}\begin{vmatrix} 4 & 1 \\ 1 & -3 \end{vmatrix}$$
$$= (1 + 6, -(4-2), -12-1)$$
$$= \boxed{(7, -2, -13)}$$

(b) For the vectors $\mathbf{u} = (-1, -2, 3), \mathbf{v} = (5, -1, 1)$, compute their dot product, determine the angle between them, and determine the magnitude of their cross product.

$$\vec{u} \cdot \vec{v} = -1 \cdot 5 + (-2) \cdot (-1) + 3 \cdot 1$$
$$= -5 + 2 + 3 = \boxed{0}$$

So $\vec{u} \perp \vec{v}$, i.e. $\boxed{\theta = \pi/2}$

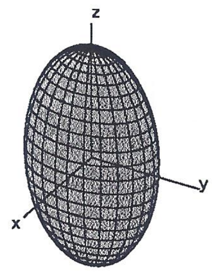
So $|\vec{u} \times \vec{v}| = |\vec{u}| \cdot |\vec{v}|$ since $\sin(\pi/2) = 1$

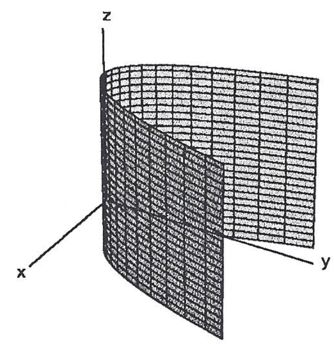
$$|\vec{u}| = \sqrt{1 + 4 + 9} = \sqrt{14}$$
$$|\vec{v}| = \sqrt{25 + 1 + 1} = \sqrt{27} = 3\sqrt{3}$$

so $\boxed{|\vec{u} \times \vec{v}| = 3\sqrt{42}}$

2. Match up the equations (I), (II), and (III) with the correct plot (a), (b), and (c).

(I) $z = 2x^2 + 3y^2$ (II) $y = x^2$ (III) $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 + \left(\frac{z}{5}\right)^2 = 1$





(a) $\boxed{\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 + \left(\frac{z}{5}\right)^2 = 1}$

(b) $\boxed{y = x^2}$

(c) $\boxed{z = 2x^2 + 3y^2}$

3. This question has 4 parts (a)–(d).

(a) Find a parametric equation for the line going through the two points

$$P = (3, 5, -1) \quad \text{and} \quad Q = (4, 8, 2).$$

$\vec{r} = \vec{r}_0 + t\vec{v}$ with $\vec{r}_0 = \vec{OP} = (3,5,-1)$

& $\vec{v} = \vec{PQ} = (4-3, 8-5, 2--1)$

$= (1,3,3)$

so $\boxed{\vec{r} = (3,5,-1) + t(1,3,3)}$

(b) What is the distance from this line to the point $S = (1, 0, 4)$?

$$\text{dist} = \frac{|\vec{PS} \times \vec{v}|}{|\vec{v}|}$$

$\vec{PS} \times \vec{v} = \vec{PS} \times \vec{PQ} = (-2,-5,5) \times (1,3,3)$

$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -2 & -5 & 5 \\ 1 & 3 & 3 \end{vmatrix} = \hat{\imath}\begin{vmatrix} -5 & 5 \\ 3 & 3 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} -2 & 5 \\ 1 & 3 \end{vmatrix} + \hat{k}\begin{vmatrix} -2 & -5 \\ 1 & 3 \end{vmatrix}$

$= (-15-15, -(-6-5), -6+5)$

$= (-30, 11, -1)$

$|\vec{PS} \times \vec{PQ}| = \sqrt{900+121+1}$

$= \sqrt{1022}$

$|\vec{v}| = \sqrt{1+9+9} = \sqrt{19}$ so $\boxed{\text{dist} = \sqrt{\frac{1022}{19}}}$

(c) Write down a (non-parametric) equation for the plane containing the triangle PQS.

$\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0$ where $\vec{r}_0 = \overrightarrow{OP} = (3,5,-1)$

$\vec{n} =$ PSxPQ $= (-30, 11, -1)$ as computed in (b)

so $-30x + 11y - z = (-30, 11, -1) \cdot (3, 5, -1)$

$= -90 + 55 + 1$

so $\boxed{-30x + 11y - z = -34}$

(d) What is the area of the triangle PQS?

$\text{Area} = \frac{1}{2} |\vec{PQ} \times \vec{PS}|$

$= \boxed{\frac{1}{2}\sqrt{1022}}$

computed in (b)

4. Consider the parametrized curve

$$\mathbf{r}(t) = (x(t), y(t)) = (t^2 + t, 3t^4 - t).$$

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the point where $t = 0$.

$$\left.\frac{dy}{dx}\right|_{t=0} = \left.\frac{dy/dt}{dx/dt}\right|_{t=0}$$

$$\frac{dx}{dt} = 2t+1$$

$$\frac{dy}{dt} = 12t^3 - 1$$

$$= \left.\frac{12t^3-1}{2t+1}\right|_{t=0}$$

$$= \frac{-1}{1}$$

$$= \boxed{-1}$$

$$\left.\frac{d^2y}{dx^2}\right|_{t=0} = \left.\frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}\right|_{t=0}$$

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{12t^3-1}{2t+1}\right)$$

$$= \frac{(2t+1)\cdot 36t^2 - 2(12t^3-1)}{(2t+1)^2}$$

$$= \left.\frac{(2t+1)(36t^2) - 2(12t^3-1)}{(2t+1)^3}\right|_{t=0}$$

$$= \frac{1\cdot 0 - 2\cdot(-1)}{(2\cdot 0+1)^3} = \boxed{2}$$

5. Consider the plane in space given by the equation $3x - 2y + 6z = 10$.

(a) Find a (non-zero) vector perpendicular to this plane.

$\boxed{\vec{n} = (3, -2, 6)}$

(b) Find a point on this plane.

let's try with $x = z = 0$, then $-2y = 10$, so $y = -5$

so $\boxed{P_0 = (0, -5, 0)}$

(c) Find the distance from this plane to the point $S = (1, 4, 5)$.

$$\text{dist} = \frac{|\vec{P_0S} \cdot \vec{n}|}{|\vec{n}|} = \frac{|(1,9,5)\cdot(3,-2,6)|}{\sqrt{9+4+36}}$$

$$= \frac{|3-18+30|}{\sqrt{49}}$$

$$= \boxed{\frac{15}{7}}$$

(d) Find a plane perpendicular to this plane.

need a normal vector $\vec{n}_1$ such that $\vec{n}_1 \cdot \vec{n} = 0$

so $(a,b,c)\cdot(3,-2,6)=0$

so $\boxed{\vec{n}_1 = (2,3,0)}$ works

OR: take $\vec{n}_1$ any vector in the plane. Can find Q in the plane by trying $y=z=0$, then

$3x=10$

so $Q = (\frac{10}{3}, 0, 0)$ is in the plane so

$\boxed{\vec{n}_1 = \vec{P_0Q} = (\frac{10}{3}, 5, 0)}$ works

Formula sheet

- Determinant of a 2×2 matrix:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

- Determinant of a 3×3 matrix:

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = u_1 \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2 \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$$

- Cross product of $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$:

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

- Projection of the vector $\mathbf{u}$ onto the vector $\mathbf{v}$:

$$\mathrm{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v}$$

- Distance from the point S to the line described by the point P_0 and the direction vector $\mathbf{v}$:

$$\text{distance} = \frac{|\overrightarrow{P_0S} \times \mathbf{v}|}{|\mathbf{v}|}$$