Midterm 2 – Math 243

Monday, March 20, 2017

This is a closed-book exam. No calculators allowed.

Justify your answers to obtain full credit (and partial credit, too).

You have 50 minutes.

This exam consists of 6 questions.

The sheets are printed on both sides. Please verify that you have all pages.

Name: Solutions **ID#:**

	Score	Out of
Question 1		
Question 2		
Question 3		
Question 4		
Question 5		
Question 6		
Total		

1. For each of the following two curves, set up, but **do no evaluate**, an integral for its arc length.

(a) In space, the parametrized curve

$$\mathbf{r}(t) = (\sin(t), 1+t, e^t + 2), \quad -1 \leq t \leq 1.$$

$\vec{r}\,'(t) = (\cos t, 1, e^t)$ $\quad |\vec{r}\,'(t)| = \sqrt{\cos^2(t) + 1^2 + e^{2t}}$

$$L = \int_{-1}^{1} |\vec{r}\,'(t)|\, dt$$

$$\boxed{L = \int_{-1}^{1} \sqrt{1 + \cos^2(t) + e^{2t}}\, dt}$$

(b) In the xy-plane, the part of the graph of $y = x^3 - x$ for x between 1 and 2.

$y = f(x) = x^3 - x$

$f'(x) = 3x^2 - 1$

$$L = \int_1^2 \sqrt{1 + f'(x)^2}\, dx$$

$$= \int_1^2 \sqrt{1 + (3x^2-1)^2}\, dx$$

$$= \int_1^2 \sqrt{1 + 9x^4 - 6x^2 + 1}\, dx$$

$$\boxed{L = \int_1^2 \sqrt{9x^4 - 6x^2 + 2}\, dx}$$

2. At time $t = 0$, a particle is at the point $(1, -1, 1)$ travelling with velocity $\mathbf{v}_0 = (1, 1, 2)$. For $t \geq 0$, a force is applied to the particle that causes its acceleration to be

$$\mathbf{a}(t) = (t, t^2, e^t).$$

Find a formula for the particle's position at time $t \geq 0$.

$$\vec{v}(t) = \int \vec{a}(t)\,dt = \int (t, t^2, e^t)\,dt$$

$$= \left(\frac{t^2}{2}, \frac{t^3}{3}, e^t\right) + \vec{c}$$

$$\vec{v}(0) = \vec{v}_0 = (1,1,2)$$

$$\left(\frac{0^2}{2}, \frac{0^3}{3}, e^0\right) + \vec{c} = (0,0,1) + \vec{c} \qquad \text{so } \vec{c} = (1,1,1)$$

$$\text{so } \vec{v}(t) = (1,1,1) + \left(\frac{t^2}{2}, \frac{t^3}{3}, e^t\right)$$

$$\vec{r}(t) = \int \vec{v}(t)\,dt = \int \left((1,1,1) + \left(\frac{t^2}{2}, \frac{t^3}{3}, e^t\right)\right)dt$$

$$= \int \left(1 + \frac{t^2}{2}, 1 + \frac{t^3}{3}, 1 + e^t\right) dt$$

$$= \left(t + \frac{t^3}{6}, t + \frac{t^4}{12}, t + e^t\right) + \vec{c}_1$$

$$\vec{r}(0) = (1,-1,1)$$

$$\left(0 + \frac{0^3}{6}, 0 + \frac{0^4}{12}, 0 + e^0\right) + \vec{c}_1 = (0,0,1) + \vec{c}_1, \quad \text{so } \vec{c}_1 = (1,-1,0)$$

$$\text{so } \boxed{\vec{r}(t) = (1,-1,0) + \left(t + \frac{t^3}{6}, t + \frac{t^4}{12}, t + e^t\right)}$$

3. Consider the curve parametrized by

$$\mathbf{r}(t) = (t^2 + t,\ \sin(t) + \cos(t),\ t + \sin(t)).$$

(a) Compute $\dfrac{d\mathbf{r}}{dt}$.

$$\boxed{\frac{d\vec{r}}{dt} = (2t+1,\ \cos(t) - \sin(t),\ 1 + \cos(t))}$$

(b) Find a parametric equation for the tangent line to this curve at the point where $t = 0$.

$\vec{r}\,'(0) = (1,\ 1-0,\ 1+1) = (1,1,2)$

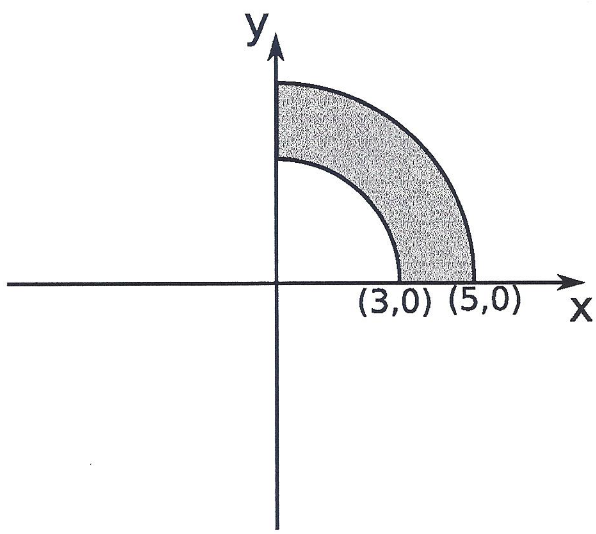
$\vec{r}(0) = (0,\ 0+1,\ 0+0) = (0,1,0)$

so $\boxed{\text{tangent line} = (0,1,0) + t(1,1,2)}$

4. In each of the following three plots (a), (b), and (c), there are some points whose Cartesian coordinates are indicated and a region is shaded. In each case, do the following two things:

   (i) write down some polar coordinates for the indicated points (no need to list all possible polar coordinates, just one will do);

   (ii) describe the region using polar coordinates.

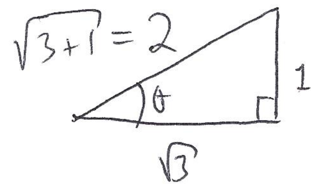


(a)

(i)

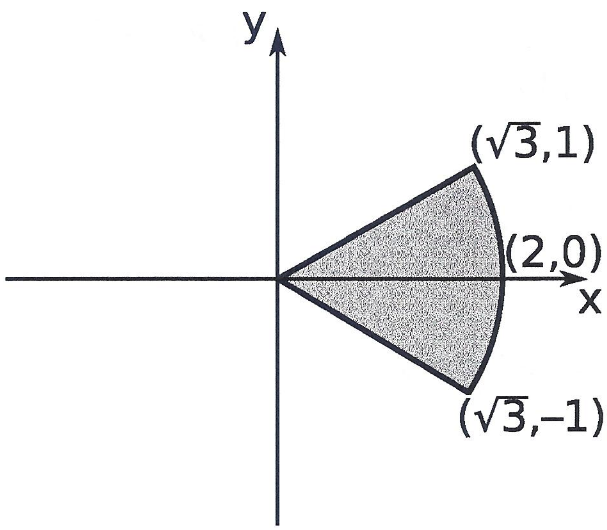
Cartesian	Polar
(3, 0)	(3, 0)
(5, 0)	(5, 0)

(ii) Region is $\boxed{\begin{array}{l} 0 \le \theta \le \pi/2 \\ 3 \le r \le 5 \end{array}}$



so $\sin\theta = \frac{1}{2}$

so $\theta = \pi/6$

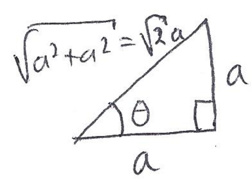


(b)

(i)

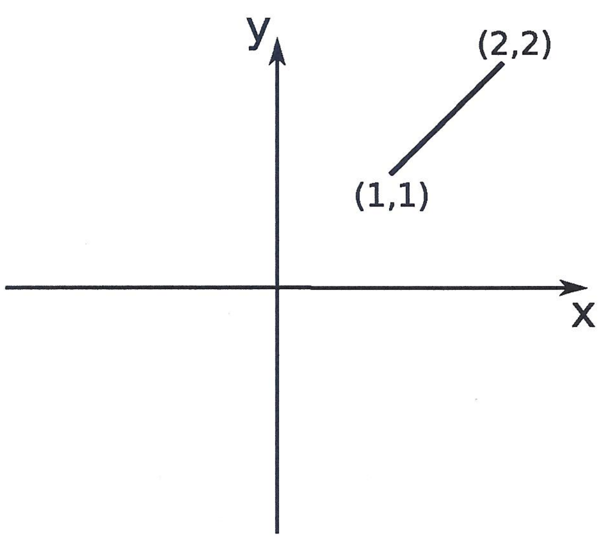
Cartesian	Polar
$(\sqrt{3}, 1)$	$(2, \pi/6)$
$(2, 0)$	$(2, 0)$
$(\sqrt{3}, -1)$	$(2, -\pi/6)$

(ii) Region is $\boxed{\begin{array}{l} -\pi/6 \le \theta \le \pi/6 \\ \& \ 0 \le r \le 2 \end{array}}$



so $\sin\theta = \frac{a}{\sqrt{2}a} = \frac{1}{\sqrt{2}}$

so $\theta = \pi/4$



(c)

(i)

Cartesian	Polar
$(1, 1)$	$(\sqrt{2}, \pi/4)$
$(2, 2)$	$(2\sqrt{2}, \pi/4)$

(ii) Region is $\boxed{\begin{array}{l} \theta = \pi/4 \\ \sqrt{2} \le r \le 2\sqrt{2} \end{array}}$

5. Find the area of the region in the xy-plane described in polar coordinates by

$$0 \le \theta \le \pi/2 \quad \text{and} \quad \theta^2 \le r \le \theta^2 + 1.$$

$f(\theta) = \theta^2$

$g(\theta) = \theta^2 + 1$

$$A = \int_0^{\pi/2} \frac{1}{2}\left(g(\theta)^2 - f(\theta)^2\right) d\theta$$

$$= \int_0^{\pi/2} \frac{1}{2}\left((\theta^2+1)^2 - (\theta^2)^2\right) d\theta$$

$$= \int_0^{\pi/2} \frac{1}{2}\left(\theta^4 + 2\theta^2 + 1 - \theta^4\right) d\theta$$

$$= \frac{1}{2}\left(\frac{2\theta^3}{3} + \theta\right)\Bigg|_0^{\pi/2}$$

$$= \frac{1}{2}\left(\frac{2}{3}\left(\frac{\pi}{2}\right)^3 + \pi/2\right)$$

$$= \frac{1}{2}\left(\frac{\pi^3}{12} + \pi/2\right)$$

$$= \boxed{\frac{\pi}{4}\left(\frac{\pi^2}{6} + 1\right)}$$

6. Find the arc length of the parametrized curve

$$\mathbf{r}(t) = \left(\sqrt{t}, \frac{t^{3/2}}{3}, \frac{(\cancel{2}\sqrt{2})t}{2}\right), \quad 0 \le t \le 1.$$

$$\vec{r}\,'(t) = \left(\frac{1}{2\sqrt{t}}, \frac{\cancel{3}\sqrt{t}}{2\cancel{3}}, \frac{\sqrt{2}}{2}\right)$$

$$|\vec{r}\,'(t)| = \sqrt{\left(\frac{1}{2\sqrt{t}}\right)^2 + \left(\frac{\sqrt{t}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2}$$

$$= \sqrt{\frac{1}{4t} + \frac{t}{4} + \frac{1}{2}}$$

$$L = \int_0^1 |\vec{r}\,'(t)|\,dt$$

$$= \int_0^1 \sqrt{\frac{1}{4t} + \frac{t}{4} + \frac{1}{2}}\,dt$$

$$= \int_0^1 \sqrt{\frac{1}{4t}(1 + t^2 + 2t)}\,dt$$

$$= \int_0^1 \frac{1}{2\sqrt{t}} \sqrt{t^2 + 2t + 1}\,dt$$

$$= \frac{1}{2}\int_0^1 \frac{1}{\sqrt{t}} \sqrt{(t+1)^2}\,dt$$

$$= \frac{1}{2}\int_0^1 \frac{t+1}{\sqrt{t}}\,dt$$

$$= \frac{1}{2}\int_0^1 \left(t^{1/2} + t^{-1/2}\right) dt$$

$$= \frac{1}{2}\left(\frac{2t^{3/2}}{3} + 2\cdot t^{1/2}\right)\Big|_0^1$$

$$= \frac{1}{2}\left(\frac{2}{3} + 2\right)$$

$$= \frac{1}{3} + 1 = \boxed{\frac{4}{3}}$$

Formula sheet

The following formulas are related to a polar curve given as $r = f(\theta)$.

- The slope of the curve:

$$\frac{dy}{dx} = \frac{f'(\theta)\sin(\theta) + f(\theta)\cos(\theta)}{f'(\theta)\cos(\theta) - f(\theta)\sin(\theta)}$$

- The arc length formula:

$$L = \int_a^b \sqrt{f(\theta)^2 + f'(\theta)^2} d\theta$$

- The area of the wedge-shaped region between the origin and $r = f(\theta)$

$$A = \int_a^b \frac{1}{2} f(\theta)^2 d\theta$$