Assignment 10 – Part 1 – Math 411

(1) Let $M = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \in M_2(\mathbf{C})$. For $v, w \in \mathbf{C}^2$, define $\langle v, w \rangle = v^{\mathrm{T}} M \overline{w}$. Show that this is an inner product on $\mathbf{C}^2$.

(2) Let $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in M_2(\mathbf{R})$. For $v, w \in \mathbf{R}^2$, define $\langle v, w \rangle = v^{\mathrm{T}} M w$. Show that this is *not* an inner product on $\mathbf{R}^2$. In particular, find a non-zero vector v such that $\langle v, v \rangle = 0$.

(3) Let V in an inner product space and let $||\cdot||$ denote the associated norm function. Show that the norm satisfies the *parallelogram law*: for all $v, w \in V$

$$||v + w||^2 + ||v - w||^2 = 2(||v||^2 + ||w||^2).$$

(4) Let $F = \mathbf{R}$ or $\mathbf{C}$ and let V be a vector space over F. A *norm on* V is a function $||\cdot|| : V \to \mathbf{R}$ such that for all $v, w \in V$ and $\lambda \in F$

(i) $||\lambda v|| = |\lambda| \cdot ||v||$,

(ii) $||v|| \geq 0$, with equality if and only if $v = 0$,

(ii) $||v + w|| \leq ||v|| + ||w||$.

Thus, the norm function associated to an inner product on V is a norm on V in this sense.

Let $V = F^n$ (for some $n \in \mathbf{Z}_{\geq 1}$). For $v = (a_1, \ldots, a_n) \in V$, define

$$||v|| = \sum_{i=1}^{n} |a_i|.$$

(a) Show that $||\cdot||$ is a norm on V.

(b) Show that it does *not* satisfy the parallelogram law of the previous exercise, and hence is not associated to an inner product. (Remark: it is in fact true that a norm on V is associated to an inner product if and only if it satisfies the parallelogram law.)