Assignment 13 – Part 1 – Math 411

(1) Let $\alpha_1, \alpha_2, \alpha_3 \in F$ and let $f(x) = x^3 + c_2x^2 + c_1x + c_0 = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)$ so that

$$c_0 = -\alpha_1\alpha_2\alpha_3, \quad c_1 = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3, \quad \text{and} \quad c_2 = -(\alpha_1 + \alpha_2 + \alpha_3).$$

Let

$$A = \begin{pmatrix} 0 & 0 & -c_0 \\ 1 & 0 & -c_1 \\ 0 & 1 & -c_2 \end{pmatrix}$$

so that, as you showed in the previous assignment, $\det(xI - A) = f(x)$. Since the α_i are the roots of $f(x)$, the kernel of $\alpha_i I - A$ is non-zero for each i. In this exercise, you'll show that these kernels are always one-dimensional.

(a) First, let's deal with some of the roots being 0. Suppose $\alpha_1 = 0$, show that $\ker(\alpha_1 I - A)$ is one-dimensional, even if either or both of α_2 and α_3 are 0. Show that the same is true if you assume $\alpha_2 = 0$ or $\alpha_3 = 0$.

(b) So, now assume none of the α_i are 0. Show that $\dim\ker(\alpha_i I - A) = 1$

(2) Let $\lambda \in F$. Let

$$A = \begin{pmatrix} \lambda & 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & 0 & \cdots & 0 \\ 0 & 0 & \lambda & 1 & & 0 \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix} \in M_n(F)$$

(i.e. that A has λ down the diagonal and 1's right above the diagonal).

(a) Show that $\det(xI - A) = (x - \lambda)^n$.

(b) Show that $\dim\ker(\lambda I - A) = 1$.

(3) A matrix $A \in M_n(F)$ is called *block diagonal* if there are matrices $A_i \in M_{n_i}(F)$, for

$i = 1, 2, \ldots, k$, such that

$$A = \begin{pmatrix} A_1 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ 0 & 0 & A_3 & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_k \end{pmatrix}.$$

For example, the matrix

$$\begin{pmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 7 & 8 \end{pmatrix}$$

is block diagonal. One writes $A = \bigoplus_{i=1}^{k} A_i$ and says that A is the *direct sum* of the A_i. Show that if A is block diagonal, then $\det(A) = \prod_{i=1}^{k} \det(A_i)$.

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