Assignment 13 – All 3 parts – Math 411

Due in the class: Friday, May 1, 2015

(1) Let $\alpha_1, \alpha_2, \alpha_3 \in F$ and let $f(x) = x^3 + c_2x^2 + c_1x + c_0 = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)$ so that

$$c_0 = -\alpha_1\alpha_2\alpha_3, \quad c_1 = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3, \quad \text{and} \quad c_2 = -(\alpha_1 + \alpha_2 + \alpha_3).$$

Let

$$A = \begin{pmatrix} 0 & 0 & -c_0 \\ 1 & 0 & -c_1 \\ 0 & 1 & -c_2 \end{pmatrix}$$

so that, as you showed in the previous assignment, $\det(xI - A) = f(x)$. Since the α_i are the roots of $f(x)$, the kernel of $\alpha_i I - A$ is non-zero for each i. In this exercise, you'll show that these kernels are always one-dimensional.

(a) First, let's deal with some of the roots being 0. Suppose $\alpha_1 = 0$, show that $\ker(\alpha_1 I - A)$ is one-dimensional, even if either or both of α_2 and α_3 are 0. Show that the same is true if you assume $\alpha_2 = 0$ or $\alpha_3 = 0$.

(b) So, now assume none of the α_i are 0. Show that $\dim\ker(\alpha_i I - A) = 1$

(2) Let $\lambda \in F$. Let

$$A = \begin{pmatrix} \lambda & 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & 0 & \cdots & 0 \\ 0 & 0 & \lambda & 1 & & 0 \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix} \in M_n(F)$$

(i.e. that A has λ down the diagonal and 1's right above the diagonal).

(a) Show that $\det(xI - A) = (x - \lambda)^n$.

(b) Show that $\dim\ker(\lambda I - A) = 1$.

(3) A matrix $A \in M_n(F)$ is called *block diagonal* if there are matrices $A_i \in M_{n_i}(F)$, for

$i = 1, 2, \ldots, k$, such that

$$A = \begin{pmatrix} A_1 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ 0 & 0 & A_3 & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_k \end{pmatrix}.$$

For example, the matrix

$$\begin{pmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 7 & 8 \end{pmatrix}$$

is block diagonal. One writes $A = \bigoplus_{i=1}^{k} A_i$ and says that A is the *direct sum* of the A_i. Show that if A is block diagonal, then $\det(A) = \prod_{i=1}^{k} \det(A_i)$.

(4) Let $n \geq 1$ and let $V = \mathbf{R}[x]_{\leq n}$ be the polynomials of degree at most n. Let $T = \frac{d}{dx}$. (You can answer all these questions by writing down the matrix for $\frac{d}{dx}$ with respect to the basis $\mathcal{B} = \{1, x, x^2, \ldots, x^n\}$.)

(a) What is the determinant of $\frac{d}{dx}$?

(b) Show that the only eigenvectors of $\frac{d}{dx}$ are the non-zero constant polynomials.

(c) What is the characteristic polynomial of $\frac{d}{dx}$?

(5) Let F be a field and let $f_1(x) = x - 2$, $f_2(x) = (x-2)^2$, and $f_3(x) = (x-2)^3$ in $F[x]$. Let

$$A_1 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}, \quad \text{and} \quad A_3 = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

(a) Show that $f_i(A_i) = 0$ but $f_j(A_i) \neq 0$ for $j < i$.

(b) Show that $\dim\ker(2I - A_1) = 3$, $\dim\ker(2I - A_2) = 2$, $\dim\ker(2I - A_3) = 1$

(c) Conclude that A_1 is diagonalizable, but A_2 and A_3 are not.

(d) Show that these three matrices have the same characteristic polynomial. What is it?

(6) (a) What are the minimal polynomials of the matrices A_1, A_2, and A_3 in the previous question?

(b) For $\frac{d}{dx} : F[x]_{\leq n} \to F[x]_{\leq n}$ as in Question (4), show that the minimal polynomial of $\frac{d}{dx}$ is x^{n+1}. Is $\frac{d}{dx}$ diagonalizable?

(7) Let $V = \mathbf{R}[x]_{\leq n}$ and let $T = x\frac{d}{dx} : V \to V$, i.e. $T(f(x)) = xf'(x)$.

(a) What are the eigenvalues of T?

(b) What is the characteristic polynomial of T?

(c) What is the minimal polynomial of T?

(d) Is T diagonalizable?