Assignment 14 – All 2 parts – Math 411

Due Friday, May 8, 2015

(1) The matrices

$$A = \begin{pmatrix} 3 & 10 & -3 & 10 \\ 1 & 5 & -1 & 3 \\ 7 & -35 & 10 & -28 \\ 1 & -12 & 3 & -8 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 3 & 3 & -1 & 3 \\ 1 & -2 & 1 & -4 \\ 7 & -35 & 10 & -28 \\ 1 & -5 & 1 & -1 \end{pmatrix}$$

have the same characteristic polynomial, namely $(x-2)^2(x-3)^2$. What are the dimensions of their eigenspaces? Are they diagonalizable? What are their minimal polynomials? What are the dimensions of their generalized eigenspaces? What are their Jordan canonical forms?

(2) (Fibonacci numbers) The Fibonacci numbers, F_n, $n \geq 1$, are defined by what we call a "second order, homogeneous, linear recurrence relation", namely

$$F_1 = 1, \quad F_2 = 1, \quad \text{and} \quad F_{n+2} = F_{n+1} + F_n, \text{ for } n \geq 1.$$

The theory of the Jordan Canonical Form provides the tools for "solving" a linear recurrence relation, i.e. for finding a formula for F_n. It goes a little something like this.

For $n \geq 1$, let $v_n = \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} \in \mathbf{R}^2$. We use a two-dimensional vector because we have a "second order" relation. We can express the recurrence relation that defines the Fibonacci numbers by

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad v_{n+1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} v_n.$$

Think about it.

(a) Suppose that $A, J, P \in M_n(F)$ and that $A = PJP^{-1}$. Show that for all $n \geq 1$, we have $A^n = PJ^nP^{-1}$.

(b) Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ be the matrix relating v_{n+1} to v_n. Find matrices $P, J \in M_2(\mathbf{R})$ such that $P^{-1}AP = J$ is diagonal (i.e. diagonalize A).

(c) Show that $v_{n+1} = A^n v_1$ and use parts (a) and (b) to write down a formula for v_{n+1} and hence for F_n.

(3) Suppose $J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ is a 2×2 Jordan block. Show that for $n \geq 1$, we have

$$J^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}.$$

(4) Define the sequence a_n by

$$a_1 = 1, \quad a_2 = 1, \quad \text{and} \quad a_{n+2} = 4a_{n+1} - 4a_n, \text{ for } n \geq 1.$$

Find a formula for a_n.

(5) Suppose A is an 8×8 matrix whose minimal polynomial is $m_A(x) = (x-1)^2(x-2)(x+7)^3$.

(a) What are the possible characteristic polynomials of A?

(b) Can A be diagonalizable? What are the possible Jordan forms of A?