Assignment 1 – All 2 parts – Math 411

Due in the class: Friday, Jan. 23, 2015

(1) Consider the following three vectors in $\mathbf{C}^2$

$$v_1 = \begin{pmatrix} 2-i \\ 5 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 3+i \end{pmatrix}, \quad w = \begin{pmatrix} 3-3i \\ 8+2i \end{pmatrix}.$$

Determine the solutions, if any, to the equation

$$\alpha_1 v_1 + \alpha_2 v_2 = w$$

with $\alpha_i \in \mathbf{C}$.

(2) Recall the following addition and multiplication tables given in class for the field $\mathbf{F}_2$ with two elements

$+$	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

Taking for granted that this gives a field, consider the following three vectors in $\mathbf{F}_2^3$

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

For each of

$$w = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

determine the solutions, if any, to the equation

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = w$$

with $\alpha_i \in \mathbf{F}_2$.

(3) Let F be a field. Show that if there is a non-zero element $\alpha \in F$ such that $\alpha + \alpha = 0$, then $\beta + \beta = 0$ for all $\beta \in F$. (Hint: first show that $1 + 1 = 0$)

(4) Show that the addition and multiplication tables in question (2) do indeed give a

field. (In class, we showed the properties of addition hold, you just need to show those of multiplication and the distributive property).

(5) Let F be a field. Show that for all a in F, we have that $(-1) \cdot a = -a$, i.e. the additive inverse of a is a times the additive inverse of 1.

(6) Consider the set of integers $\mathbf{Z}$ and recall that $a \in \mathbf{Z}$ is called *even* if there is a number $b \in \mathbf{Z}$ such that $a = 2b$; otherwise, a is called *odd*. Consider the following relation, denoted $\equiv$, on the set $\mathbf{Z}$: $a \equiv b$ if a and b are either both even or both odd.

   (a) Go through the explanation, which will likely feel quite silly, that shows that $\equiv$ is an *equivalence* relation.

   (b) Given $a, b \in \mathbf{Z}$, we say that a *divides* b if there is a number $d \in \mathbf{Z}$ such that $b = ad$. For $a, b \in \mathbf{Z}$, show that $a \equiv b$ if and only if 2 divides $a - b$.

(7) Sudoku-ish: There is a field with three elements $0, 1$, and some other element, let's say α. It's called $\mathbf{F}_3$. Write down what its addition and multiplication tables must be (in class, we did this for $\mathbf{F}_2$). (You don't need to verify that those tables actually give a field, you just have to figure out what the axioms of a field force those tables to be). Provide your reasoning.

(8) Do the same thing as the previous question, but with 4 elements $0, 1, \alpha, \beta$. (Hint: like in harder Sudoku problems, you may have to make a guess at some point while filling out the addition table, in which case you'll have to see why only one of the possibilities works out).