Assignment 2 – Parts 1 & 2 – Math 411

(1) A matrix A is said to be in reduced row echelon form (RREF) if it is in row echelon form and it satisfies the following two properties:

(RREF1) Every entry lying above a leading entry is 0.

(RREF2) Every leading entry is 1.

This exercise will walk you through the proof that every $m \times n$ matrix A is row-equivalent to an $m \times n$ matrix that is in reduced row echelon form.

(a) Suppose A is in row echelon form. Explain why doing any elementary row operation of type (III) (i.e. $r_i \rightsquigarrow \alpha r_i$, $\alpha \neq 0$) leaves A in row echelon form. Explain why doing an elementary row operation of type (II) sending $r_i \rightsquigarrow r_i + \alpha r_j$ leaves A in row echelon form if $i < j$.

(b) Use induction on the number of rows of A to show that A is row-equivalent to a matrix in RREF. (Hint: first, let A' be a matrix in REF that is row-equivalent to A. Then, proceed similarly to how we did in class for row echelon form and deal with the first row and use induction for the remaining rows. Part (a) of this question will be relevant, so don't forget to explain how it matters.)

(2) Suppose A is an $m \times n$ matrix in RREF and B is an $m \times k$ matrix in RREF. Show that the $m \times (n + k)$ matrix obtained by placing B to the right of A, in notation $(A\ B)$, is in RREF.

(3) Let F be any field. Write down all RREF 2×2 matrices over F.

(4) Let $F = \mathbf{F}_2$ be the field with two elements. Write down all REF 2×3 matrices over F. Circle the ones that are in RREF.

(5) In class, we defined a polynomial over the field F to be an "expression" of the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ where $n \in \mathbf{Z}_{\geq 0}$, $a_i \in F$, and I said x^i is just a "symbol" used as a "place-holder". You may be used to thinking of a polynomial as just a function. This exercise will tell you why in general you shouldn't think of polynomials as functions. Here's what happens. The set of all polynomials over F (in the "variable" x) is denoted $F[x]$. Given a polynomial

$$f(x) = \sum_{i=0}^{n} a_i x^i \in F[x],$$

one gets a function from F to itself which we will denote $\widetilde{f}$ in this exercise (such a function is called a "polynomial function"). Specifically, $\widetilde{f} : F \to F$ is the function which takes as input $\alpha \in F$ and outputs

$$\widetilde{f}(\alpha) = \text{“}f(\alpha)\text{”} = \sum_{i=0}^{n} a_i \alpha^i$$

(i.e. what you probably expect it to be). Now, here's the the thing. The polynomials $f(x) = x$ and $g(x) = x^2$ are not equal: the first one has its first coefficient non-zero, while the second one has its second coefficient non-zero. But, show that if $F = \mathbf{F}_2$, the field with two elements, then $f(x)$ and $g(x)$ give the same function from F to itself (i.e. $\widetilde{f} = \widetilde{g}$ even though $f \neq g$).

(6) Let $V = \mathrm{Func}(\mathbf{Z}_{\geq 0}, F)$ be the vector space of functions from $\mathbf{Z}_{\geq 0}$ to F. Let W be the subset of functions that are non-zero at only finitely many values (so, for instance, the constant function that sends every non-negative integer to 1 is *not* in W, but the function that sends the first n non-negative integers to 1 and all the rest to 0 is in W, for any n). Show that W is a subspace of V.

(7) (a) Show that for any field F, there is a map $\varphi : \mathbf{Z} \to F$ such that

(i) $\varphi(0) = 0$,

(ii) $\varphi(1) = 1$,

(iii) $\varphi(m + n) = \varphi(m) + \varphi(n)$,

(iv) $\varphi(mn) = \varphi(m)\varphi(n)$.

(Hint: let φ be the function defined by

- $\varphi(0) = 0$,
- $\varphi(1) = 1$,
- for $n > 0$, $\varphi(n) = \overbrace{1 + 1 + \cdots + 1}^{n \text{ times}}$,
- for $n > 0$, $\varphi(-n) = -\varphi(n)$,

and show that it satisfies the properties.)

(b) For $F = \mathbf{F}_2$, show that $\varphi(m)$ is 0 is m is even and 1 if m is odd.

(c) Let V be a vector space and let w be a vector in V. Let $\langle w \rangle = \{\alpha \cdot w : \alpha \in F\}$ be the set of all scalar multiples of w. Show that $\langle w \rangle$ is a subspace of V.

(8) Determine which of the following W are subspaces of the given V. Justify your answer.

(a) For $n \geq 2$, $V = F^n$, $W = \{(a_1, \ldots, a_n) \in F^n : a_2 = 2a_1\}$.

(b) For $n \geq 2$, $V = F^n$, $W = \{(a_1, \ldots, a_n) \in F^n : a_2 = a_1 + 1, \text{ for } 1 \leq j \leq n\}$.

(c) For $n \geq 2$, $V = F^n$, $W = \{(a_1, \ldots, a_n) \in F^n : a_j = ja_1, \text{ for } 1 \leq j \leq n\}$.

(d) $V = F^2$, $W = \{(a_1, a_2) \in F^2 : a_2 = a_1^2\}$. (Hint: this depends F!)

(e) $V = \mathrm{Func}(F, F)$, $W = \{f : F \to F : f(1) = 0\}$

(f) $V = \mathrm{Func}(F, F)$, $W = \{f : F \to F : f(1) = 1\}$