Assignment 3 – Part 1 – Math 411

(1) Show that a "smallest" (the technical word is "minimal") generating set is indeed linearly independent. I.e. suppose that $S \subseteq V$ satisfies the following two properties:

  (i) $\mathrm{Span}(S) = V$, and

  (ii) for any $S' \subseteq V$ with the property that $\mathrm{Span}(S') = V$, we have that $S' \not\subsetneq S$.

  Show that S is linearly independent.

(2) Dually[1], show that a maximal linearly independent set is a generating set. First, translate this statement into two properties of a set S as I did in the first question, then prove the statement.

(3) In the next lecture, we will discuss the notion of a *partial order* on a set X. A partial order, often denoted $\leq$, on the set X is a relation on X satisfying the following three conditions: for all $x, y, z \in X$,

  (i) $x \leq x$ (reflexivity),

  (ii) if $x \leq y$ and $y \leq x$, then $x = y$ (antisymmetry),

  (iii) if $x \leq y$ and $y \leq z$, then $x \leq z$.

  (a) Let $X = \mathbf{Z}_{\geq 1}$ and recall that if $a, b \in \mathbf{Z}_{\geq 1}$, we say that a divides b if there is a $d \in \mathbf{Z}_{\geq 1}$ such that $b = ad$. Since we already use $\leq$ to mean something for integers, let's use a different notation for this new relation: say that $a \mid b$ if a divides b (this is actually the standard notation). Show that $\mid$ is a partial order on $\mathbf{Z}_{\geq 1}$.

  (b) A *minimal element* for a partial order $\leq$ on X is an element $x \in X$ such that there is no $y \in X$ with $y < x$ (here $y < x$ means $y \leq x$ and $y \neq x$). What are the minimal elements of the partial order $\mid$ on $\mathbf{Z}_{\geq 1}$.

  (c) If $Y \subseteq X$ and $\leq$ is a partial order on X, you can *restrict* $\leq$ to a partial order on Y (just say that for $y, y' \in Y$, $y \leq y'$ if that is true when thinking of y, y' as being elements of X). What are the minimal elements of the restriction of $\mid$ to $\mathbf{Z}_{\geq 2}$.

[1]The word 'dual' is used a lot in mathematics. Here, a generating set is like a big set and a linearly independent set is like a small set. We say that big and small are dual notions. The first question asks you to show that a minimal "big" set is "small", while the second question asks you to show 'dually' that a maximal "small" set is "big".