Assignment 3 – All 3 parts – Math 411

Due in the class: Friday, Feb. 6, 2015

(1) Show that a "smallest" (the technical word is "minimal") generating set is indeed linearly independent. I.e. suppose that $S \subseteq V$ satisfies the following two properties:

(i) $\mathrm{Span}(S) = V$, and

(ii) for any $S' \subseteq V$ with the property that $\mathrm{Span}(S') = V$, we have that $S' \not\subsetneq S$.

Show that S is linearly independent.

(2) Dually[1], show that a maximal linearly independent set is a generating set. First, translate this statement into two properties of a set S as I did in the first question, then prove the statement.

(3) In the next lecture, we will discuss the notion of a *partial order* on a set X. A partial order, often denoted $\leq$, on the set X is a relation on X satisfying the following three conditions: for all $x, y, z \in X$,

(i) $x \leq x$ (reflexivity),

(ii) if $x \leq y$ and $y \leq x$, then $x = y$ (antisymmetry),

(iii) if $x \leq y$ and $y \leq z$, then $x \leq z$.

(a) Let $X = \mathbf{Z}_{\geq 1}$ and recall that if $a, b \in \mathbf{Z}_{\geq 1}$, we say that a divides b if there is a $d \in \mathbf{Z}_{\geq 1}$ such that $b = ad$. Since we already use $\leq$ to mean something for integers, let's use a different notation for this new relation: say that $a \mid b$ if a divides b (this is actually the standard notation). Show that $\mid$ is a partial order on $\mathbf{Z}_{\geq 1}$.

(b) A *minimal element* for a partial order $\leq$ on X is an element $x \in X$ such that there is no $y \in X$ with $y < x$ (here $y < x$ means $y \leq x$ and $y \neq x$). What are the minimal elements of the partial order $\mid$ on $\mathbf{Z}_{\geq 1}$.

(c) If $Y \subseteq X$ and $\leq$ is a partial order on X, you can *restrict* $\leq$ to a partial order on Y (just say that for $y, y' \in Y$, $y \leq y'$ if that is true when thinking of y, y' as being elements of X). What are the minimal elements of the restriction of $\mid$ to $\mathbf{Z}_{\geq 2}$.

[1] The word 'dual' is used a lot in mathematics. Here, a generating set is like a big set and a linearly independent set is like a small set. We say that big and small are dual notions. The first question asks you to show that a minimal "big" set is "small", while the second question asks you to show 'dually' that a maximal "small" set is "big".

(4) Let V be a vector space.

(a) Show that if $S \subseteq V$ contains the zero vector, then it is linearly dependent.

(b) Suppose $S \subseteq V$ is linearly independent. Show that any subset of S is linearly independent.

(c) Suppose $S \subseteq V$ is generates V. Show that any superset of S is generating set.

(d) Let $\mathcal{B}$ be a basis of V. Show that any proper subset of $\mathcal{B}$ is not a generating set and any proper superset of $\mathcal{B}$ is linearly dependent.

(5) Let $V = \mathrm{Func}(\mathbf{R}, \mathbf{R})$. Let $S = \{1, \sin(x), \cos(x)\} \subseteq V$. Show that S is linearly independent. Is $\{1, \sin^2(x), \cos^2(x)\}$ linearly independent?

(6) Let V be a vector space and let L be a linearly independent subset. Show that a minimal generating set containing L is a basis. (This is a slight strengthening of question (1).)

(7) Let F be a field and let $V = F^3$. Consider the vectors

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad w_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, w_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, w_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

(a) Show that $\{v_1, v_2, v_3\}$ and $\{w_1, w_2, w_3\}$ are both linearly independent.

(b) Go through the steps of the Steinitz substitution lemma to show that for $j = 1, 2, 3$, you can relabel v_1, v_2, v_3 so that

$$\{w_1, \ldots, w_j, v_{j+1}, \ldots v_3\}$$

is still a basis of F^3.

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