Assignment 4 – Parts 1 & 2 – Math 411

(1) Let $F = \mathbf{F}_2$ and let V be a two-dimensional vector space over F.

(a) How many vectors are there in V?

(b) How many subspaces of V are there?

(c) How many bases of V are there?

(d) How many bases of a three-dimensional vector space over F are there?

(e) How many bases of a two-dimensional vector space over $\mathbf{F}_3$ are there?

(2) Let F be a field and let $m, n \in \mathbf{Z}_{\geq 1}$. Let $M_{m,n}(F)$ denote the set of $m \times n$ matrices over F (i.e. with entries in F). For two matrices $A = (a_{ij})$ and $B = (b_{ij})$ in $M_{m,n}(F)$, let $A + B = (a_{ij} + b_{ij})$ be the usual matrix addition and $\alpha A = (\alpha a_{ij})$ be the usual scalar multiplication with $\alpha \in F$. This makes $M_{m,n}(F)$ into a vector space over F (no need to show this, though you can think about why).

(a) Find a basis for $M_{m,n}(F)$.

(b) What is the dimension of $M_{m,n}(F)$?

(3) Let $V = M_{n,n}(F)$ and recall that if $A = (a_{i,j})$ is a matrix, then its *transpose* is the matrix A^T whose (i, j)-entry is $a_{j,i}$. A matrix $A \in M_{n,n}(F)$ is called *symmetric* if $A^T = A$ and *antisymmetric* if $A^T = -A$.

(a) Let $\mathcal{S}$ and $\mathcal{A}$ be the set of symmetric and antisymmetric matrices in $M_{n,n}(F)$, respectively. Show that $\mathcal{S}$ and $\mathcal{A}$ are subspaces of $M_{n,n}(F)$.

(b) Show that $M_{n,n}(F) = \mathcal{S} \oplus \mathcal{A}$.

(c) A matrix $A = (a_{i,j})$ is called *strictly upper-triangular* if $a_{i,j} = 0$ whenever $j \leq i$, i.e. if all the elements on the diagonal and below are zero. Let $\mathcal{U}$ be the set of strictly upper-triangular matrices in $M_{n,n}(F)$. Is $\mathcal{U}$ a subspace of $M_{n,n}(F)$? If so, is $M_{n,n}(F) = \mathcal{S} \oplus \mathcal{U}$? Is $M_{n,n}(F) = \mathcal{A} \oplus \mathcal{U}$?

(4) Let W_1, W_2 be two planes through the origin in $\mathbf{R}^3$. Show that $\dim(W_1 \cap W_2) \geq 1$.

(5) Let $V = \mathrm{Func}(\mathbf{R}, \mathbf{R})$. Let

$$\mathcal{P} = \{f : \mathbf{R} \to \mathbf{R} : f(x) \geq 0 \text{ for all } x\} \text{ and } \mathcal{N} = \{f : \mathbf{R} \to \mathbf{R} : f(x) \leq 0 \text{ for all } x\}.$$

Are $\mathcal{P}$ and $\mathcal{N}$ subspaces of V? If so, show that $V = \mathcal{P} \oplus \mathcal{N}$.

(6) Let V be a finite-dimensional vector space and let $W_i \leq V$ for $i \in I$ be such that

$$V = \bigoplus_{i \in I} W_i.$$

(a) Show that I is finite.

(b) Show that, as claimed in class, $\dim V = \sum_{i \in I} \dim W_i$.

(7) Let V be a vector space and let $\mathcal{B}$ be a basis of V. For each $b \in \mathcal{B}$, let $W_b = \mathrm{Span}(\{b\}) \leq V$. Go through the details of proving that

$$V = \bigoplus_{b \in \mathcal{B}} W_b,$$

as claimed in class.