Assignment 6 – All 2 parts – Math 411

Due in the class: Friday, Feb. 27, 2015

(1) Let

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 1 & 2 & 2 & 7 \\ 3 & 6 & 2 & 9 \end{pmatrix}.$$

(a) Consider $A \in M_{3,4}(\mathbf{R})$ and find a basis for its column space, its row space, and its null space.

(b) Considering A to be in $M_{3,4}(\mathbf{C})$, do the same.

(c) What about in $M_{3,4}(\mathbf{Q})$?

(d) What about in $M_{3,4}(\mathbf{F}_2)$? (Here, you should think of 7, for instance, as $1+1+1+1+1+1+1$, which is 1 in $\mathbf{F}_2$. Viewing an integer as an element of $\mathbf{F}_2$ was one of the points of Question (7) of Assignment 2).

(2) A (finite) sequence of vectors $v_1, \ldots, v_m \in F^n$ is said to be in *echelon form* if the matrix whose rows are $v_1, \ldots, v_m$ is in REF. We'll say they are in *reduced* echelon form if the corresponding matrix is in RREF. If $W \leq F^n$, a basis that is in echelon form will be called an *echelon basis*; if it is in reduced echelon form, we'll call it a *reduced* echelon basis. Let

$$v_1 = \begin{pmatrix} 1 \\ -3 \\ -3 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 3 \\ 9 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 3 \\ -3 \\ 3 \\ 0 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 4 \\ -6 \\ 0 \\ 1 \end{pmatrix}.$$

Let $W = \mathrm{Span}(v_1, v_2, v_3, v_4)$.

(a) Find a subset of $\{v_1, v_2, v_3, v_4\}$ that forms a basis of W.

(b) Find a reduced echelon basis of W.

(3) Let $A \in M_{m,n}(F)$ and let A' be any REF of A.

(a) Show that the locations of the leading entries of A' (i.e. the (i, j)-coordinates of the leading entries) are the same as the locations of the leading entries (i.e. pivots) of the RREF(A). (Hint: think about what row operations need to be done to go from A' to RREF(A).)

(b) Show that the non-zero rows of A' are a basis of the row space of A.

(c) This is just a remark: this means that you don't need to bring A into RREF to determine a basis of the column space or the row space; just finding any REF of A will do.

(4) (a) Let $A \in M_{m,n}(F)$ and recall that $\text{Nul}(A)$ is the space of solutions of the system $(A \quad 0)$. In class, we explained how to write down a basis $v_1, \ldots, v_d$ of the null space of A (where d was the number of free variables). Specifically, if $j_1 < j_2 < \cdots < j_d$ are the indices of the non-pivot columns of $\text{RREF}(A)$ and

$$i(k) = k - \ell(k)$$

(where $\ell(k)$ denotes the maximum index of j_ℓ such that $j_\ell \leq k$, or 0 if $k < j_1$), then the kth entry of v_t was

$$\begin{cases} 1 & \text{if } k = j_t, \\ 0 & \text{if } k = j_r \text{ with } r \neq t, \\ 0 & \text{if } k > j_t, \\ -a'_{i(k)j_t} & \text{otherwise,} \end{cases}$$

where $a'_{i,j}$ is the (i,j)-entry of $\text{RREF}(A)$. (Note that this is not what I had in class, that was a bit off). Show that, reversing the order, $v_d, v_{d-1}, \ldots, v_1$ is a reduced echelon basis of $\text{Nul}(A)$. (Hint: You don't actually need the $i(k)$ business to answer this question, I just included that to correct what I said in class.)

(b) Find a matrix whose null space is spanned by the vectors v_1, v_2, v_3, v_4 of Question (2).

(5) Let $F = \mathbf{R}$ and let $V = \mathbf{R}$. For $\alpha \in F$, recall that we had the "multiplication by α" linear transformation $T_\alpha : V \to V$ sending v to $\alpha \cdot v$. Show that all linear transformations from from V to V are of this form.

(6) Let F be any field and let $A = (a_{ij}) \in M_{m,n}(F)$. For

$$b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \in F^n,$$

we defined in class the function $T_A : F^n \to F^m$ sending b to Ab, where Ab is the vector in F^m whose ith entry is

$$(Ab)_i = \sum_{k=1}^{n} A_{ik} b_k.$$

We claimed that T_A is linear. Prove this claim.

(7) Prove the following basic properties of linear transformations that we stated in class. Let $T : V \to W$ be a linear transformation.

(i) $T(0) = 0$.

(ii) $T(-v) = -T(v)$.

(iii) If $V' \leq V$, then $T|_{V'} : V' \to W$ is linear.

(iv) For all $V' \leq V$, $T(V') \leq W$.

(v) The kernel of T is a subspace of V.

(vi) Given a set of vectors $\{v_i : i \in I\}$ in V, if $\{T(v_i) : i \in I\}$ is linearly independent, then so is $\{v_i : i \in I\}$.

(vii) Given a set of vectors $\{v_i : i \in I\}$ in V, if $\{v_i : i \in I\}$ spans V, then $\{T(v_i) : i \in I\}$ spans the image of T.