Assignment 7 – Part 1 – Math 411

(1) (a) Let $T' : U \to V$ and $T : V \to W$ be two linear transformations between vector spaces over F. Show that the composition $T \circ T' : U \to W$ is a linear transformation.

(b) Let $T_i : V \to W$, for $i = 1, 2$, be two linear transformations between V and W. Define a new function called $T_1 + T_2$ from V to W by

$$(T_1 + T_2)(v) := T_1(v) + T_2(v) \quad \text{for } v \in V.$$

Show that $T_1 + T_2$ is linear.

(c) Let $T : V \to W$ linear. For a scalar $\lambda \in F$, define a new function $\lambda T : V \to W$ by

$$(\lambda T)(v) := \lambda \cdot T(v) \quad \text{for } v \in V.$$

Show that λT is linear.

(2) For two vector spaces V and W over F, let $\mathrm{Hom}(V, W)$ denote the set of all linear transformations from V to W. In the previous question, we defined addition and scalar multiplication operations on this set. Define the 0 function from V to W by sending every $v \in V$ to $0 \in W$. Show that with this structure $\mathrm{Hom}(V, W)$ is a vector space over F. (Note: the book calls this $L(V, W)$, the 'L' being for 'linear'; I used 'Hom' for 'homomorphism', because that's the more general name for a linear transformation and this is accordingly a more general notation.)

(3) (a) Let X be a non-empty set and let W be a vector space. For a non-empty subset Y of X, define the function $T_Y : \mathrm{Func}(X, W) \to \bigoplus_{y \in Y} W$ by $T_Y(f) = (f(y))_{y \in Y}$ (where $(f(y))_{y \in Y}$ denotes the tuple whose 'yth' entry is $f(y)$). Show that T_Y is linear.

(b) Consider the following functions in $\mathrm{Func}(\mathbf{R}, \mathbf{R})$: $f_0(x) = 1$, $f_1(x) = \cos(x)$, and $f_2(x) = \cos(2x)$. Show that $\{f_0, f_1, f_2\}$ is linearly independent.