

Assignment 7 – Parts 1 & 2 – Math 411

- (1) (a) Let $T' : U \rightarrow V$ and $T : V \rightarrow W$ be two linear transformations between vector spaces over F . Show that the composition $T \circ T' : U \rightarrow W$ is a linear transformation.
- (b) Let $T_i : V \rightarrow W$, for $i = 1, 2$, be two linear transformations between V and W . Define a new function called $T_1 + T_2$ from V to W by

$$(T_1 + T_2)(v) := T_1(v) + T_2(v) \quad \text{for } v \in V.$$

Show that $T_1 + T_2$ is linear.

- (c) Let $T : V \rightarrow W$ linear. For a scalar $\lambda \in F$, define a new function $\lambda T : V \rightarrow W$ by

$$(\lambda T)(v) := \lambda \cdot T(v) \quad \text{for } v \in V.$$

Show that λT is linear.

- (2) For two vector spaces V and W over F , let $\text{Hom}(V, W)$ denote the set of all linear transformations from V to W . In the previous question, we defined addition and scalar multiplication operations on this set. Define the 0 function from V to W by sending every $v \in V$ to $0 \in W$. Show that with this structure $\text{Hom}(V, W)$ is a vector space over F . (Note: the book calls this $L(V, W)$, the ‘ L ’ being for ‘linear’; I used ‘Hom’ for ‘homomorphism’, because that’s the more general name for a linear transformation and this is accordingly a more general notation.)
- (3) (a) Let X be a non-empty set and let W be a vector space. For a non-empty subset Y of X , define the function $T_Y : \text{Func}(X, W) \rightarrow \bigoplus_{y \in Y} W$ by $T_Y(f) = (f(y))_{y \in Y}$ (where $(f(y))_{y \in Y}$ denotes the tuple whose ‘ y th’ entry is $f(y)$). Show that T_Y is linear.
- (b) Consider the following functions in $\text{Func}(\mathbf{R}, \mathbf{R})$: $f_0(x) = 1$, $f_1(x) = \cos(x)$, and $f_2(x) = \cos(2x)$. Show that $\{f_0, f_1, f_2\}$ is linearly independent.
- (4) Which of the following linear transformations are injective? Which are surjective? Justify your answers.

- (a) Let F be any field. Consider $\frac{d}{dx} : F[x] \rightarrow F[x]$, where $\frac{d}{dx} \left(\sum_{i=0}^d a_i x^i \right) = \sum_{i=1}^d i a_i x^{i-1}$.

(b) For $F = \mathbf{R}$, consider $R_\theta : \mathbf{R}^2 \rightarrow \mathbf{R}^2$, rotation counterclockwise around the origin by θ .

(c) For $F = \mathbf{R}$. The linear transformation $T_A : \mathbf{R}^3 \rightarrow \mathbf{R}^3$ given by $T_A(v) = Av$, where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 7 & 8 \end{pmatrix}.$$

(d) For $F = \mathbf{R}$. The linear transformation $T_A : \mathbf{R}^3 \rightarrow \mathbf{R}^3$ given by $T_A(v) = Av$, where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 5 & 8 \end{pmatrix}.$$

(e) For $F = \mathbf{R}$, consider $I : \mathcal{C}(\mathbf{R}, \mathbf{R}) \rightarrow \mathbf{R}$ given by $I(f) = \int_0^1 f(x)dx$.

(5) (a) Is there a linear transformation $T : \mathbf{R}^3 \rightarrow \mathbf{R}[x]$ such that

$$T \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = x+2, \quad T \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = x^2-1, \quad T \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix} = 2x, \quad \text{and} \quad T \begin{pmatrix} 6 \\ 0 \\ 2 \end{pmatrix} = x^2+1?$$

(b) Is there a linear transformation $T : \mathbf{R}^3 \rightarrow \mathbf{R}[x]$ such that

$$T \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = x+2 \quad \text{and} \quad T \begin{pmatrix} 6 \\ 0 \\ 2 \end{pmatrix} = x^2+1?$$

(6) Which of the following pairs of vector spaces are isomorphic? For those that are, write down an isomorphism. Justify your answers.

(a) $V = F^4$ and $W = M_{2,2}(F)$.

(b) Let $\frac{d}{dx} : F[x] \rightarrow F[x]$ be the differentiation map. Let $V = F[x]$ and $W = \text{im}(\frac{d}{dx})$.