Assignment 7 – All 3 parts – Math 411

Due in the class: Monday, Mar. 9, 2015

(1) (a) Let $T' : U \to V$ and $T : V \to W$ be two linear transformations between vector spaces over F. Show that the composition $T \circ T' : U \to W$ is a linear transformation.

(b) Let $T_i : V \to W$, for $i = 1, 2$, be two linear transformations between V and W. Define a new function called $T_1 + T_2$ from V to W by

$$(T_1 + T_2)(v) := T_1(v) + T_2(v) \quad \text{for } v \in V.$$

Show that $T_1 + T_2$ is linear.

(c) Let $T : V \to W$ linear. For a scalar $\lambda \in F$, define a new function $\lambda T : V \to W$ by

$$(\lambda T)(v) := \lambda \cdot T(v) \quad \text{for } v \in V.$$

Show that λT is linear.

(2) For two vector spaces V and W over F, let $\mathrm{Hom}(V, W)$ denote the set of all linear transformations from V to W. In the previous question, we defined addition and scalar multiplication operations on this set. Define the 0 function from V to W by sending every $v \in V$ to $0 \in W$. Show that with this structure $\mathrm{Hom}(V, W)$ is a vector space over F. (Note: the book calls this $L(V, W)$, the 'L' being for 'linear'; I used 'Hom' for 'homomorphism', because that's the more general name for a linear transformation and this is accordingly a more general notation.)

(3) (a) Let X be a non-empty set and let W be a vector space. For a non-empty subset Y of X, define the function $T_Y : \mathrm{Func}(X, W) \to \bigoplus_{y \in Y} W$ by $T_Y(f) = (f(y))_{y \in Y}$ (where $(f(y))_{y \in Y}$ denotes the tuple whose 'yth' entry is $f(y)$). Show that T_Y is linear.

(b) Consider the following functions in $\mathrm{Func}(\mathbf{R}, \mathbf{R})$: $f_0(x) = 1$, $f_1(x) = \cos(x)$, and $f_2(x) = \cos(2x)$. Show that $\{f_0, f_1, f_2\}$ is linearly independent.

(4) Which of the following linear transformations are injective? Which are surjective? Justify your answers.

(a) Let F be any field. Consider $\frac{d}{dx} : F[x] \to F[x]$, where $\frac{d}{dx}\left(\sum_{i=0}^{d} a_i x^i\right) = \sum_{i=1}^{d} i a_i x^{i-1}$.

(b) For $F = \mathbf{R}$, consider $R_\theta : \mathbf{R}^2 \to \mathbf{R}^2$, rotation counterclockwise around the origin by θ.

(c) For $F = \mathbf{R}$. The linear transformation $T_A : \mathbf{R}^3 \to \mathbf{R}^3$ given by $T_A(v) = Av$, where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 7 & 8 \end{pmatrix}.$$

(d) For $F = \mathbf{R}$. The linear transformation $T_A : \mathbf{R}^3 \to \mathbf{R}^3$ given by $T_A(v) = Av$, where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 5 & 8 \end{pmatrix}.$$

(e) For $F = \mathbf{R}$, consider $I : \mathcal{C}(\mathbf{R}, \mathbf{R}) \to \mathbf{R}$ given by $I(f) = \int_0^1 f(x)dx$.

(5) (a) Is there a linear transformation $T : \mathbf{R}^3 \to \mathbf{R}[x]$ such that

$$T\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = x+2, \quad T\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = x^2-1, \quad T\begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix} = 2x, \quad \text{and} \quad T\begin{pmatrix} 6 \\ 0 \\ 2 \end{pmatrix} = x^2+1?$$

(b) Is there a linear transformation $T : \mathbf{R}^3 \to \mathbf{R}[x]$ such that

$$T\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = x+2 \quad \text{and} \quad T\begin{pmatrix} 6 \\ 0 \\ 2 \end{pmatrix} = x^2+1?$$

(6) Which of the following pairs of vector spaces are isomorphic? For those that are, write down an isomorphism. Justify your answers.

(a) $V = F^4$ and $W = M_{2,2}(F)$.

(b) Let $\frac{d}{dx} : F[x] \to F[x]$ be the differentiation map. Let $V = F[x]$ and $W = \mathrm{im}(\frac{d}{dx})$.

(7) Let $T : \mathbf{R}^4 \to \mathbf{R}^4$ be given by the formula

$$T(x, y, z, t) = (x + y + z + t, x + z, x + t - z, x + y + 2t - z).$$

Let $\mathcal{B} = \{e_1, \ldots, e_4\}$ be the standard basis of $\mathbf{R}^4$.

(a) Determine the matrix ${}_{\mathcal{B}}[T]_{\mathcal{B}}$ of T with respect to $\mathcal{B}$ and $\mathcal{C}$?.

(b) Let $\mathcal{C} = \{e_1 + e_2, e + 2 + e_3, e + 3 + e_4, e_4\}$. This is a basis of $\mathbf{R}^4$. Determine the matrix ${}_{\mathcal{C}}[T]_{\mathcal{B}}$.

(c) Determine the matrix ${}_{\mathcal{B}}[T]_{\mathcal{C}}$.

(d) Is T injective? Is it surjective? Find bases for its kernel and its image.

(8) Let $I : \mathbf{R}[x]_{\leq d} \to \mathbf{R}[x]_{\leq d+1}$ be the linear transformation that sends $f(x) \in \mathbf{R}[x]$ to $\displaystyle\int_0^x f(t)dt$, i.e.

$$\sum_{i=0}^{d} a_i x^i \mapsto \sum_{i=1}^{d} a_i \frac{x^{i+1}}{i+1}.$$

Let $\mathcal{B} = \{1, x, x^2, \ldots, x^d\}$ be the standard basis of $\mathbf{R}[x]_{\leq d}$ and let $\mathcal{C} = \{1, x, x^2, \ldots, x^{d+1}\}$ be the standard basis of $\mathbf{R}[x]_{\leq d+1}$.

(a) Determine the matrix ${}_{\mathcal{C}}[I]_{\mathcal{B}}$ of I with respect to $\mathcal{B}$ and $\mathcal{C}$.

(b) Is I injective? Is it surjective? Find bases for its kernel and its image.

(c) Let $\mathcal{B}' = \{1, 1 + x, 1 + x + x^2, \ldots, 1 + x + x^2 + \cdots + x^d\}$. This is a basis of $\mathbf{R}[x]_{\leq d}$. Determine the matrix ${}_{\mathcal{C}}[I]_{\mathcal{B}'}$.

(d) Let $\mathcal{C}' = \{1, 1 + x, 1 + x + x^2, \ldots, 1 + x + x^2 + \cdots + x^{d+1}\}$. This is a basis of $\mathbf{R}[x]_{\leq d+1}$. Determine the matrix ${}_{\mathcal{C}'}[I]_{\mathcal{B}}$?