Assignment 8 – Parts 1, 2, & 3 – Math 411

(1) Let V be a vector space over F, let $\alpha \in F$, and let $T_\alpha : V \to V$ be the linear transformation $v \mapsto \alpha v$. Prove the following properties we discussed in class.

(a) $T_0 = 0$

(b) $T_1 = 1$

(c) $T_{\alpha_1+\alpha_2} = T_{\alpha_1} + T_{\alpha_2}$

(d) $T_{\alpha_1\alpha_2} = T_{\alpha_1}T_{\alpha_2}$

(Remark: these show that F can be thought of as a "subalgebra" of $\mathrm{End}(V)$.)

(2) Let $T : V \to W$ be a linear transformation. Prove the following statements (without assuming V and W are finite-dimensional).

(a) If $\dim(V) < \dim(W)$, then T cannot be surjective.

(b) If $\dim(V) > \dim(W)$, then T cannot be injective.

(3) Let $V = \mathbf{R}[x]_{\leq 3}$ be the $\mathbf{R}$-vector space of polynomials over $\mathbf{R}$ of degree at most 3 and let $\mathcal{B} := \{1, x, (3x^2 - 1)/2, (5x^3 - 3x)/2\}$, so that $\mathcal{B}$ is a basis of V.

(a) Find the matrix ${}_{\mathcal{B}}[I]_{\mathcal{B}}$ of the bilinear form

$$I(f(x), g(x)) = \int_{-1}^{1} f(x)g(x)dx.$$

(b) What is the matrix of I with respect to the standard basis $\{1, x, x^2, x^3\}$ of V.

(4) For $a < b$ in $\mathbf{R}$, let $V = \mathcal{C}([a, b], \mathbf{R})$ be the $\mathbf{R}$-vector space of continuous functions from the closed interval $[a, b]$ to $\mathbf{R}$ and let $w(x)$ be a fixed function in V. Define the following function $I : V \times V \to \mathbf{R}$

$$I(f(x), g(x)) = \int_{a}^{b} f(x)g(x)w(x)dx.$$

(a) Show that $I(x)$ is a bilinear form.

(b) Take $a = 0$ and $b = \pi$, let $f_n(x) = \cos(nx)$ for $n = 0, 1, 2$, and let $W = \mathrm{Span}(f_0, f_1, f_2) \leq V$. Consider I as a bilinear form $W \times W \to \mathbf{R}$ and find the matrix of I with respect to the basis $\{f_0, f_1, f_2\}$ of W.