Assignment 9 – Part 1 – Math 411

(1) Let $A = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$. Let $B : F^3 \times F^2 \to F$ be the bilinear form given by $B(v,w) = v^{\mathrm{T}}Aw$. Is B non-degenerate? Is the map $v \mapsto B_v$ (where $B_v(w) = B(v,w)$ for $w \in F^2$) injective? What about the map $w \mapsto B_w$ (where $B_w(v) = B(v,w)$ for $v \in F^3$) injective? What if A is the matrix $\begin{pmatrix} 1 & 2 \\ 3 & 6 \\ 4 & 8 \end{pmatrix}$?

(2) Let $m, n \in \mathbf{Z}_{\geq 1}$ and let $B : F^m \times F^n \to F$ be a bilinear form. We saw that there is an $m \times n$ matrix A such that $B(v,w) = v^{\mathrm{T}}Aw$.

- (a) For $v \in F_m$, recall that we had a linear function $B_v \in \mathrm{Hom}(F^n, F)$ given by $B_v(w) = B(v,w)$. Show that B_v is represented by the row vector $v^{\mathrm{T}}A$.
- (b) Similarly, for $w \in F^n$, we have $B_w \in \mathrm{Hom}(F^m, F)$ given by $B_w(v) = B(v,w)$. Show that B_w is represented by the row vector $(Aw)^{\mathrm{T}}$.
- (c) Show that if B is non-degenerate, then $m = n$.
- (d) Given that $m = n$, show that B is non-degenerate if and only if $\mathrm{Nul}(A) = 0$.

(3) Let V and W be finite-dimensional vector spaces and let $B : V \times W \to F$ be a non-degenerate bilinear form.

- (a) Show that $\dim(V) = \dim(W)$ and the maps $v \mapsto B_v$ and $w \mapsto B_w$ are isomorphisms $V \xrightarrow{\sim} W^*$ and $W \xrightarrow{\sim} V^*$, respectively.
- (b) Let $\mathcal{C} = \{c_1, \ldots, c_n\}$ be a basis of V. Show that there is a unique bilinear form $B : V \times V \to F$ such that $B(c_i, c_j) = \delta_{ij}$ and that B is non-degenerate. (Hint: in fact, for any V and W and any bases $\mathcal{C}$ and $\mathcal{C}'$ of V and W, respectively, and any choice of elements $\alpha_{c,c'} \in F$, there is a unique bilinear form $B : V \times W \to F$ such that $B(c, c') = \alpha_{c,c'}$).
- (c) With the same notation as in part (b), show that the isomorphism $v \mapsto B_v$ from V to V^* given by B is the same as the isomorphism we gave in class from V to V^* sending c_i to c_i^*.