**Midterm – Math 411**

**Friday, March 6, 2015**

This is a closed-book exam. No calculators allowed.

**Justify your answers** to obtain full credit (and partial credit, too).

You have 50 minutes.

This exam consists of 4 questions.

The sheets are printed on both sides. Please verify that you have all pages.

**Name:** Solutions **ID#:**

| | Score | Out of |
|---|---|---|
| Question 1 | | |
| Question 2 | | |
| Question 3 | | |
| Question 4 | | |
| Total | | |

1. Let $A \in M_{4,5}(\mathbf{R})$ be given by

$$A = \begin{pmatrix} 1 & -1 & 3 & 1 & 8 \\ 2 & -2 & 4 & 4 & 13 \\ 3 & -3 & 7 & 5 & 24 \\ 2 & -2 & 4 & 4 & 16 \end{pmatrix}.$$

(a) Find a basis for the row space of $A$ and a basis for the column space of $A$.

(b) What is the dimension of the null space of $A$?

(c) Let $V$ be the polynomials over $\mathbf{R}$ of degree at most 4 with basis

$$\mathcal{B} = \{1, 1+x, 1+x^2, 1+x^3, 1+x^4\}$$

and let $W$ be the polynomials over $\mathbf{R}$ of degree at most 3 with basis

$$\mathcal{C} = \{1+x, x+x^2, x^2+x^3, x^3\}.$$

Suppose $T : V \to W$ is a linear transformation whose matrix ${}_{\mathcal{C}}[T]_{\mathcal{B}}$ with respect to $\mathcal{B}$ and $\mathcal{C}$ is the matrix $A$. Is $T$ injective?

(d) Let $T$ be as in part (c). What is $T(x^3)$?

(a) $\begin{pmatrix} 1 & -1 & 3 & 1 & 8 \\ 2 & -2 & 4 & 4 & 13 \\ 3 & -3 & 7 & 5 & 24 \\ 2 & -2 & 4 & 4 & 16 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -1 & 3 & 1 & 8 \\ 0 & 0 & -2 & 2 & -3 \\ 0 & 0 & -2 & 2 & 0 \\ 0 & 0 & -2 & 2 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -1 & 3 & 1 & 8 \\ 0 & 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

so $\boxed{\text{basis of } Row(A) = \left\{ \begin{pmatrix} 1 \\ -1 \\ 3 \\ 1 \\ 8 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -2 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -3 \end{pmatrix} \right\}}$, $\boxed{\text{basis of } Col(A) = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 7 \\ 4 \end{pmatrix}, \begin{pmatrix} 8 \\ 13 \\ 24 \\ 16 \end{pmatrix} \right\}}$

(b) $null(A) = 5 - 3 = \boxed{2}$

(c) $\boxed{\text{No}:}$ $\dim \ker(T) = null(A) = 2 > 0$.

(d) $x^3 = (1+x^3) - 1$ so $[T(x^3)]_{\mathcal{C}} = [T(1+x^3)]_{\mathcal{C}} - [T(1)]_{\mathcal{C}} = \begin{pmatrix} 1 \\ 4 \\ 5 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 2 \\ 2 \end{pmatrix}$

So $T(x^3) = 0 \cdot (1+x) + 2 \cdot (x+x^2) + 2 \cdot (x^2+x^3) + 2x^3$
$= \boxed{2x + 4x^2 + 4x^3}$

2. In each case, determine if the set $S$ is or is not a subspace of $V$. Justify.

(a) Let $T_1$ and $T_2$ be two linear transformations $V \to W$ (where $V$ and $W$ are any two given vector spaces). Let $S := \{v \in V : T_1(v) = T_2(v)\}$.

(b) Let $V = M_{m,n}(F)$ be the $m \times n$ matrices over a field $F$ (where $m, n \in \mathbf{Z}_{\geq 1}$) and let $S := \{A \in V : \mathrm{Nul}(A) = 0\}$.

(a) let $v_1, v_2 \in V$, then $T_1(v_1+v_2) = T_1(v_1) + T_1(v_2) = T_2(v_1) + T_2(v_2) = T_2(v_1+v_2)$ ✓

let $\lambda \in F$, $v \in V$, then $T_1(\lambda v) = \lambda T_1(v) = \lambda T_2(v) = T_2(\lambda v)$ ✓

so $\boxed{\text{Yes}}$

(b) $\boxed{\text{No}}$ : The zero matrix has $\mathrm{Nul}(0) = F^n \neq 0$. So $0 \notin S$ so $S$ is not a subspace

3. Let $V$ be a finite-dimensional vector space of dimension $n$ over a field $F$.

(a) Let $W_0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_m$ be a finite sequence of subspaces $W_i$ of $V$ with $W_{i+1}$ strictly bigger than $W_i$. Show that $m \leq n$.

(b) In fact, show that there is always a sequence $W_0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_n$ of $n+1$ subspaces of $V$ (in other words, the dimension of $V$ is the biggest $m$ such that there is a strictly increasing sequence of subspaces $W_0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_m$).

(a) Slick proof: $W_i$ are finite-dimensional so $W_{i+1} \supsetneq W_i \Rightarrow \dim(W_{i+1}) \geq \dim(W_i) + 1$

so $\dim(W_m) \geq m$

but $\dim(W_m) \leq \dim(V) = n$ so $m \leq n$ QED

(b) let $B = \{b_1, b_2, \ldots, b_n\}$ be a basis of $V$ & let $W_0 = 0$ & $W_i = \text{Span}\{b_1, b_2, \ldots, b_i\}$ for $i = 1, \ldots, n$

Then the $W_i$ are subspaces.

Claim: $W_{i+1} \supsetneq W_i$

proof: $W_{i+1} \supseteq W_i$ by definition so check that $\exists\, v \in W_{i+1}$ with $v \notin W_i$

well $b_{i+1} \notin W_i$ because $b_{i+1} \in W_i = \text{Span}\{b_1, \ldots, b_i\}$ implies $b_1, \ldots, b_{i+1}$ is lin. dep. but it's a subset of a basis so it's lin. indep. QED

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(a) Less slick proof: let $v_i \in W_{i+1} \setminus W_i$ for $i = 1, \ldots, m$.

Claim: $\{v_1, \ldots, v_m\}$ is lin indep.

proof: let $i_0$ be least $i$ s.t. $v_i$ is a lin comb of $\{v_1, \ldots, v_{i-1}\}$.

Then $v_{i_0} \in \text{Span}\{v_1, \ldots, v_{i_0 - 1}\} \subseteq W_{i_0 - 1}$. But $v_{i_0} \notin W_{i_0 - 1}$. contradiction

In an n-dim'l vect. space, every lin. indep set has $\leq n$ vectors.

so $m \leq n$. QED

4. Let $V$ be a vector space and let $V_1$ and $V_2$ be two subspaces such that $V = V_1 \oplus V_2$.

(a) Show that there is a basis $\mathcal{B}$ of $V$ such that $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ where $\mathcal{B}_i$ is a basis of $V_i$.

(b) Let $T_i : V_i \to V_i$ be a linear transformation. Show that there is a unique linear transformation $T : V \to V$ such that $T(v_i) = T_i(v_i)$ for all $v_i \in V_i$.

(a) Slick proof: let $L =$ a basis $B_1$ of $V_1$ & let $G = L \cup V_2$. Then $L$ is lin. indep. and $G$ is a generating set

By Theorem in class, $\exists$ a basis $B$ of $V$ s.t. $L = B_1 \subseteq B \subseteq L \cup V_2 = G$

let $B_2 = B \setminus B_1$.

Claim: $B_2$ is a basis of $V_2$. ($B_2$ is lin. indep, so just show $\mathrm{Span}(B_2) = V_2$)

proof: $\forall b \in B_2$, $b \notin B_1$, but $b \in G$ so $b \in V_2$ so $\mathrm{Span}\{b : b \in B_2\} \subseteq V_2$

To show $V_2 \subseteq \mathrm{Span}(B_2)$: let $v \in V_2 \subseteq V$, $B$ is a basis of $V$

so $\exists \alpha_b, \beta_b \in F$ st. $v = \sum_{b \in B_1} \alpha_b b + \sum_{b \in B_2} \beta_b b$.

Claim: $\alpha_b = 0 \ \forall b \in B_1$

proof: every $w \in V$ can be written as $w = v_1 + v_2$ for unique $v_i \in V_i$

now an $v \in V_2$ can be written as $v = 0 + v$ $\quad 0 \in V_1$, $v \in V_2$

so $\sum_{b \in B_1} \alpha_b b \in V_1$ & $\sum_{b \in B_2} \beta_b b \in V_2$

implies $\sum_{b \in B_1} \alpha_b b = 0$. $B_1$ lin. indep implies $\alpha_b = 0 \ \forall b \in B_1$

QED.

so $v = \sum_{b \in B_2} \beta_b b$ so $V_2 \subseteq \mathrm{Span}(B_2)$. QED

QED

(b) By part (a), $\exists$ basis $B = B_1 \cup B_2$ of $V$ s.t. $B_i$ is a basis of $V_i$. For $b \in B$, let

$$W_b = \begin{cases} T_1(b) & \text{if } b \in B_1 \\ T_2(b) & \text{if } b \in B_2 \end{cases}$$

By Theorem in class, $\exists!\ T : V \to V$ s.t. $T(b) = W_b \ \forall b \in B$.

Claim: $\forall v_i \in V_i$, $T(v_i) = T_i(v_i)$

proof: We need to show $T|_{V_i} = T_i$.

Well, for $b \in B_i$, $T|_{V_i}(b) = T_i(b)$ by definition.

so $T|_{V_i}$ & $T_i$ agree on a basis, so they are equal. QED