Math 411
Asst. 1 Solutions

(1) Row reduce

$$\left(\begin{array}{cc|c} 2-i & 1 & 3-3i \\ 5 & 3+i & 8+2i \end{array}\right) \xrightarrow{R_1 \to R_1\cdot(2+i)} \left(\begin{array}{cc|c} 5 & 2+i & 9-3i \\ 5 & 3+i & 8+2i \end{array}\right)$$

$$\xrightarrow{R_2 \to R_2 - R_1} \left(\begin{array}{cc|c} 5 & 2+i & 9-3i \\ 0 & 1 & -1+5i \end{array}\right) \xrightarrow{R_1-(2+i)R_2 \to R_1} \left(\begin{array}{cc|c} 5 & 0 & 16-12i \\ 0 & 1 & -1+5i \end{array}\right)$$

$(3-3i)(2+i) = (6+3)+i(3-6)$

$(-1+5i)\cdot(2+i) = (-2-5)+i(10-1) = -7+9i$

so unique sol'n: $\boxed{\alpha_1 = \frac{16}{5} - \frac{12}{5}i \qquad \alpha_2 = -1+5i}$

(2)

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right) \xrightarrow{R_2 \to R_2+R_1} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right) \xrightarrow{R_3 \to R_3+R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

no solutions!

If augmented column is $\begin{pmatrix}0\\0\\0\end{pmatrix}$, left-hand part is unchanged so get

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

so α_3 is a "free variable"

$\alpha_1 = -\alpha_3$

$\alpha_2 = -\alpha_3$

of course α_3 can only be 0 or 1 so there are two solutions

$$\boxed{\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \text{ or } \begin{pmatrix}1\\1\\1\end{pmatrix}}$$

(3) if $\alpha \neq 0$ & $\alpha+\alpha=0$, then $\alpha\cdot(1+1)=0$ so $\underbrace{\alpha^{-1}\cdot\alpha\cdot(1+1)}_{(1+1)} = \underbrace{\alpha^{-1}\cdot 0}_{0}$ so $1+1=0$

so $\forall \beta \in F$, $\beta+\beta = \beta\cdot(1+1) = \beta\cdot 0 = 0$

QED

(4) Assoc. of $\cdot$: if one of $a, b,$ or c is 0, both sides are 0 ✓

if none are zero, then $1\cdot(1\cdot 1) \stackrel{?}{=} (1\cdot 1)\cdot 1$

$$\underset{1\cdot 1=1}{1\cdot(1\cdot 1)} = 1 = \underset{1 = 1\cdot 1}{(1\cdot 1)\cdot 1}$$ ✓

Comm. of $\cdot$: ~~If one of a,b~~ the table is symmetric ✓

Mult. identity: $1\cdot 0=0$, $1\cdot 1=1$

Mult. inverse: $1\cdot 1=1$

Distr.: if $a=0$, both sides are 0.

$1\cdot(b+c)=$ ~~...~~ $b+c=1\cdot b+1\cdot c$ ✓ QED

(5) $\underbrace{(1+(-1))\cdot a}_{0\cdot a=0} = 1\cdot a+(-1)\cdot a = a+(-1)\cdot a$

so $(-1)\cdot a=-a$ (since $a+(-1)\cdot a=0$ & the additive inverse is unique)

(6) (a) reflexivity: $a\equiv a$ since a has the same parity as itself

symmetry: if $a\equiv b$, then a & b have the same ", so b & a do, too! so $b\equiv a$

transitivity: if a & b have the same parity & b & c do as well, then a & c do!

(b) $a\equiv b$ iff 2 divides $a-b$:

($\Rightarrow$) if $a\equiv b$, then either $a=2a'$ & $b=2b'$, so that $a-b=2a'-2b'=2(a'-b')$

or $a=2a'+1$ & $b=2b'+1$, so that $a-b=2a'-2b'=2(a'-b')$

in either case, 2 divides $a-b$.

($\Leftarrow$) (by contrapositive) if $a\not\equiv b$, then one of a or b is $2k$, the other is $2m+1$ for $k,m\in\mathbb{Z}$

so $a-b=2d\pm 1$ for $d=k-m$ or $m-k$

so 2 does not divide $a-b$ QED

(7)

$+$	0	1	α
0	0	1	α
1	1	α	0
α	α	0	1

If $1+1=0$, then, to avoid repetition, $1+\alpha=\alpha$, but then $1+0=0$ ie $1=0$.

~~1+1 ≠ 0 or else ... (by question (3))~~

so $1+1=\alpha$. Necessarily, $1+\alpha=0$ since rows (& cols) can't repeat an element. Similarly the last entry is $\alpha+\alpha=1$

(7) (cont'd)

$\cdot$	0	1	α
0	0	0	0
1	0	1	α
α	0	α	1

to avoid repetition, must have $\alpha \cdot \alpha = 1$

(8)

+	0	1	α	β
0	0	1	α	β
1	1	0	β	α
α	α	β	0	1
β	β	α	1	0

~~If If~~ If $1+1=\alpha$, then $\alpha+1=\beta$ otherwise $\beta+\beta=\beta$, which implies $\beta=0$. But then $\beta+\beta=0$. By (3), this means $a+a=0$ $\forall a \in \mathbb{F}_4$.
Similarly, if $1+1=\beta$. Thus $a+a=0$ $\forall a \in \mathbb{F}_4$.
Then $1+\alpha=\beta$ (since $1+\alpha=\alpha \Rightarrow 1=0$)
This leaves that $\alpha+\beta=1$

$\cdot$	0	1	α	β
0	0	0	0	0
1	0	1	α	β
α	0	α	β	1
β	0	β	1	α

If $\alpha \cdot \alpha = 1$, then to avoid repetition, $\alpha \cdot \beta = \beta$ so $\alpha = 1$. Thus $\alpha \cdot \alpha = \beta$
& $\alpha \cdot \beta = 1$
& $\beta \cdot \beta = \alpha$