Math 411

Asst. 10 solutions

(1) This is not an inner product. Indeed, $\left\langle \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix} \right\rangle = (1\ 0)\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$

$= (0\ i)\begin{pmatrix}1\\0\end{pmatrix} = 0.$

(2) $\left\langle \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix} \right\rangle = (1\ 0)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = (0\ 1)\begin{pmatrix}1\\0\end{pmatrix} = 0$. but $\begin{pmatrix}1\\0\end{pmatrix} \neq \begin{pmatrix}0\\0\end{pmatrix}$ so this is not an inner product.

(3) $\|v+w\|^2 + \|v-w\|^2 = \langle v+w, v+w\rangle + \langle v-w, v-w\rangle$

$= (\langle v,v\rangle + \cancel{\langle w,v\rangle} + \cancel{\langle v,w\rangle} + \langle w,w\rangle) + (\langle v,v\rangle - \cancel{\langle w,v\rangle} - \cancel{\langle v,w\rangle} + \langle w,w\rangle)$

$= 2(\langle v,v\rangle + \langle w,w\rangle) = 2(\|v\|^2 + \|w\|^2)$

(4) (a)(i) $\|\lambda v\| = \sum_{i=1}^{n} |\lambda a_i| = \sum_{i=1}^{n} |\lambda|\cdot|a_i| = |\lambda| \sum_{i=1}^{n} |a_i| = |\lambda|\cdot\|v\|$

(ii) If $v \neq 0$, $\exists j$ st $a_j \neq 0$. so $|a_j| > 0$. Also, $\forall i$, $|a_i| \geq 0$. So $\|v\| = \sum_{i=1}^{n} |a_i| \geq 0$ & if $v \neq 0$, it is $\geq |a_j| > 0$.

(iii) $\forall \alpha,\beta \in \mathbb{R}$ or $\mathbb{C}$, $|\alpha+\beta| \leq |\alpha| + |\beta|$. so $\|v+w\| = \sum_{i=1}^{n} |a_i+b_i| \leq \sum_{i=1}^{n} (|a_i| + |b_i|) = \sum |a_i| + \sum |b_i|$

$= \|v\| + \|w\|$

QED

(b) let $v = (1,0,0,\dots,0)$ & $w = (\frac{1}{2}, \frac{3}{2}, 0, 0, \dots 0)$, so $v+w = (\frac{3}{2}, \frac{3}{2}, 0,0,0\dots)$, $v-w = (\frac{1}{2}, -\frac{3}{2}, 0, 0 \dots 0)$

Then $\|v+w\|^2 + \|v-w\|^2 = (\frac{3}{2} + \frac{3}{2})^2 + (\frac{1}{2} + \frac{3}{2})^2 = 3^2 + 2^2 = 13$

$2(\|v\|^2 + \|w\|^2) = 2(1 + (\frac{1}{2} + \frac{3}{2})^2) = 2\cdot(1+2^2) = 10 \neq 13.$

(In fact, it looks like (v,w) gives an example of violation of the parallelogram law iff $\|proj_w v\| < \|w\|$ & $\|proj_v w\| < \|v\|$).

(5)(a) Let $v_1 = 1$, $v_2 = x$, $v_3 = x^2$, $v_4 = x^3$.

$w_1 = v_1 = \boxed{1}$ $\quad \langle w_1, w_1\rangle = \int_{-1}^{1} 1\,dx = 2$ $\qquad \langle v_2, w_1\rangle = \int_{-1}^{1} x\,dx = 0$ (odd function)

$w_2 = v_2 - proj_{w_1} v_2 = v_2 - \frac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1 = v_2 = \boxed{x}$ $\qquad \langle w_2, w_2\rangle = \int_{-1}^{1} x^2 dx = 2\int_0^1 x^2 dx = 2\frac{x^3}{3}\Big|_0^1 = \frac{2}{3}$

$w_3 = v_3 - \frac{\langle v_3, w_1\rangle}{\langle w_1, w_1\rangle} w_1 - \frac{\langle v_3, w_2\rangle}{\langle w_2, w_2\rangle} w_2$. $\quad \langle v_3, w_2\rangle = \int_{-1}^{1} x^3 dx = 0$ (odd function)

(5)(a)(cont'd) $\langle v_3, w_1\rangle = \int_{-1}^{1} x^2\,dx = \frac{2}{3}$

so $w_3 = x^2 - \frac{2/3}{2}\cdot 1 = \boxed{x^2 - \frac{1}{3}}$

$w_4 = v_4 - \frac{\langle v_4, w_1\rangle}{\langle w_1, w_1\rangle} w_1 - \frac{\langle v_4, w_2\rangle}{\langle w_2, w_2\rangle} w_2 - \frac{\langle v_4, w_3\rangle}{\langle w_3, w_3\rangle} w_3$

$= x^3 - \frac{2/5}{2/3} x = \boxed{x^3 - \frac{3}{5} x}$

$\langle v_4, w_1\rangle = \int_{-1}^{1} x^3\,dx = 0$

$\langle v_4, w_3\rangle = \int_{-1}^{1}\left(x^5 - \frac{x^3}{3}\right)dx = 0$

$\langle v_4, w_2\rangle = \int_{-1}^{1} x^4\,dx = 2\int_0^1 x^4\,dx$

$= 2\frac{x^5}{5}\Big|_0^1 = \frac{2}{5}$

so $\boxed{B = \{1, x, x^2 - \frac{1}{3}, x^3 - \frac{3}{5}x\}}$ is an orthogonal basis of Span(S)

(Comparing to Question (3) of Asst 8, the first two are the same, the third is $\frac{2}{3}$ of that of Asst 8, & the last one is $\frac{2}{5}$ of that of asst 8.)

(b) One can see that $[x^2]_B = \begin{pmatrix} \frac{1}{3} \\ 0 \\ 1 \\ 0 \end{pmatrix}$

(c) let $v_1 = 1$, $v_2 = x$, $v_3 = x^2$

$w_1 = v_1 = \boxed{1}$ $\quad \langle w_1, w_1\rangle = \int_0^\infty e^{-x}\,dx = -e^{-x}\Big|_0^\infty = -e^{-\infty} - -e^0 = \boxed{1}$

$w_2 = v_2 - \frac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1$, $\quad \langle v_2, w_1\rangle = \int_0^\infty x e^{-x}\,dx = -xe^{-x}\Big|_0^\infty + \int_0^\infty e^{-x}\,dx = 0 + 1$

$u = x \quad dv = e^{-x}\,dx$

$du = dx \quad v = -e^{-x}$

since $\lim_{x\to\infty} \frac{x}{e^x} = 0$

$= \boxed{x - 1}$

$w_3 = v_3 - \frac{\langle v_3, w_1\rangle}{\langle w_1, w_1\rangle} w_1 - \frac{\langle v_3, w_2\rangle}{\langle w_2, w_2\rangle} w_2$,

$= x^2 - 2\cdot 1 - 4(x-1)$

$= \boxed{x^2 - 4x + 2}$

$\langle v_3, w_1\rangle = \langle v_2, w_2\rangle = \int_0^\infty x^2 e^{-x}\,dx = \frac{-x^2}{e^x}\Big|_0^\infty + 2\int_0^\infty x e^{-x}\,dx$

$u = x^2 \quad dv = e^{-x}dx$

$du = 2x\,dx \quad v = -e^{-x}$

$= 0 + 2$

$\boxed{B = \{1, x-1, x^2 - 4x + 2\}}$

$\langle v_3, w_2\rangle = \int_0^\infty x^2(x-1)e^{-x}\,dx = \int_0^\infty x^3 e^{-x}\,dx - \int_0^\infty x^2 e^{-x}\,dx$

$0 = \frac{-x^3}{e^{-x}}\Big|_0^\infty + 3\int x^2 e^{-x}\,dx$

$u = x^3 \quad dv = e^{-x}dx$

$du = 3x^2 \quad v = -e^{x}$

$= 6$

$6 - 2 = 4$

(d) so $\boxed{[x^2]_B = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}}$

$\langle w_2, w_2\rangle = \int_0^\infty (x^2 - 2x + 1)e^{-x}\,dx = 2 - 2\cdot 1 + 1 = \boxed{1}$