Math 411

Asst. 11 Solutions

(1)(a) $\Leftarrow$ Suppose $v \in V$ is s.t. $\langle v, s\rangle = 0 \ \forall s \in S$. Let $w \in W = \mathrm{Span}(S)$. Then $\exists\, \alpha_s \in F$ s.t. $w = \sum_{s\in S} \alpha_s s$. So $\langle v, w\rangle = \langle v, \sum_{s\in S} \alpha_s s\rangle = \sum_{s \in S} \alpha_s \langle v, s\rangle = \sum_{s\in S} \alpha_s 0 = 0$

$\Rightarrow$ Suppose $v \in V$ is s.t. $\langle v, w\rangle = 0 \ \forall w \in W$. Since $S \subseteq W$, this means $\langle v, s\rangle = 0 \ \forall s \in S$. QED

(b) Let $W = \mathrm{Col}(A)$. Then $u \in V$ is orthogonal to all $w \in W$ iff $\langle u, c\rangle = 0$ $\forall$ columns c of A, by part (a). Let $c_1, \dots, c_n$ be the cols of A. Then u is $\perp$ to all $w \in W$ iff $c_i \cdot u = c_i^T u = 0$ $\forall i$. i.e. $A^T u = 0$.

The orthogonal proj of v onto W is the vector $w \in W$ s.t. $v - w \perp w'$ $\forall w' \in W$ so it is $w \in W$ s.t. $A^T(v - w) = 0$. $W = \mathrm{Col}(A)$ so the w we seek is of the form $w = A\hat{x}$ for some $\hat{x} \in \mathbb{R}^n$. If $\hat{x} \in \mathbb{R}^n$ is st. $A^T(v - A\hat{x}) = 0$, then $A\hat{x} \in W$ & $(v - A\hat{x}) \perp w'$ $\forall w' \in W$ so $A\hat{x} = \mathrm{proj}_W v$. QED

(c) (This implies $m \geq n$ otherwise, the columns can't be lin. indep.)
Consider the linear system $(A^TA)x = 0$. We want to show only $x = 0$ is a solution. Suppose $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ is a sol'n, & recall that $Ax = x_1c_1 + x_2c_2 + \dots + x_nc_n$ so $(A^TA)x = 0$ means $A^T(x_1c_1 + \dots + x_nc_n) = 0$, i.e. there's a vector $w = x_1c_1 + \dots x_nc_n$ in $W = \mathrm{Col}(A)$ st. $A^T w = 0$. But by part (b), this means w is orthogonal to all $w' \in W = \mathrm{Col}(A)$. In particular, take $w' = w$, then $\langle w, w\rangle = 0$, so $w = 0$.
So $Ax = 0$. A has lin. indep columns, so $Ax = 0$ has only $x = 0$ as a solution. QED

(d) Let $\hat{x} = (A^TA)^{-1}A^T v$. We must show that $A^T(v - A\hat{x}) = 0$ (by part (b)).
Well, $A^TA\hat{x} = A^TA(A^TA)^{-1}A^Tv = A^Tv$
so $A^Tv - A^TA\hat{x} = A^Tv - A^Tv = 0$. QED.

(2)(a) ~~$\|v-w'\| = \|(v-w)+(w-w')\| \leq \|v-w\|$~~

$\|v-w'\|^2 = \langle v-w', v-w'\rangle = \langle (v-w)+(w-w'), (v-w)+(w-w')\rangle = \langle v-w, v-w\rangle + \langle w-w', v-w\rangle + \langle v-w, w-w'\rangle + \langle w-w', w-w'\rangle$

now $\langle v-w, w''\rangle = 0 \ \forall w'' \in W$, & $w - w' \in W$

so $= \|v-w\|^2 + \underbrace{\|w-w'\|^2}_{\geq 0}$ so $\|v-w\|^2 \leq \|v-w'\|^2$ QED.

(2) (b) let $w' \in W$. let $w'' \in W$. let $\tilde{w} = w' - w'' \overset{\text{closure of subtraction.}}{\in} W$. Then $w'' = w' - \tilde{w}$ QED

(c) As in part (a), $\|v-w'\|^2 = \|v-w\|^2 + \underbrace{\langle w-w', v-w\rangle + \langle v-w, w-w'\rangle}_{= 2Re(\langle v-w, w-w'\rangle)} + \langle w-w', w-w'\rangle$

~~so~~ now $\langle w-w', v-w\rangle = \overline{\langle v-w, w-w'\rangle}$ so

If $\|v-w'\| \geq \|v-w\| \ \forall w' \in W$, Then $\forall w'' \in W$, write $w'' = w-w'$ for some $w' \in W$ (by part (b))

Then $\|v-w'\|^2 - \|v-w\|^2 \geq 0$

$$= 2Re(\langle v-w, w''\rangle) + \langle w'', w''\rangle \quad \text{QED}$$

(d) set $w'' = \dfrac{-\langle v-w, w-w'\rangle}{\langle w-w', w-w'\rangle}(w-w')$ in part (c),

Then $\langle v-w, w''\rangle = -\dfrac{\overline{\langle v-w, w-w'\rangle}}{\langle w-w', w-w'\rangle}\langle v-w, w-w'\rangle = -\dfrac{|\langle v-w, w-w'\rangle|^2}{\langle w-w', w-w'\rangle} \in \mathbb{R}$

& $\langle w'', w''\rangle = \dfrac{|\langle v-w, w-w'\rangle|^2}{\langle w-w', w-w'\rangle^{\cancel{2}}} \cancel{\langle w-w', w-w'\rangle}$

so $2Re(\langle v-w, w''\rangle) + \langle w'', w''\rangle \geq 0$

$$= -2\frac{|\langle v-w, w-w'\rangle|^2}{\langle w-w', w-w'\rangle} + \frac{|\langle v-w, w-w'\rangle|^2}{\langle w-w', w-w'\rangle} = \underbrace{-\overbrace{\frac{|\langle v-w, w-w'\rangle|^2}{\langle w-w', w-w'\rangle}}^{\geq 0}}_{\leq 0} \geq 0$$

so ≥ 0 iff $|\langle v-w, w-w'\rangle| = 0$

i.e. $\langle v-w, w-w'\rangle = 0 \quad \forall w' \in W$

so by part (b), $\langle v-w, \tilde{w}\rangle = 0 \ \forall \tilde{w} \in W$. QED

Summary:

Part (a) shows that if $v-w$ is orthogonal to all $w' \in W$, Then w is a best approx to v in W

Parts (c) & (d) together show that if $\bar{v}$ is a best approx to v in W, Then $\forall w' \in W$, $\langle v-\bar{v}, w'\rangle = 0$.

(e) Suppose $w, w' \in W$ are such that $v-w$ & $v-w'$ are both orthogonal to all vectors in W.

We want to show that $w = w'$, ~~Consider $\langle v-w$~~ i.e. $\langle w-w', w-w'\rangle = 0$

Consider $\langle w-w', w-w'\rangle = \underbrace{\langle v-w, w-w'\rangle}_{=0} + \langle w-w', w-w'\rangle = \langle v-w+w-w', w-w'\rangle$

$= \langle v-w', w-w'\rangle$. Now $w-w' \in W$

so $= 0$. QED

(3) (a) Suppose $A\hat{x} = p (= \mathrm{proj}_W b)$, $W = \mathrm{Col}(A)$. Then $p = A\hat{x} \in \mathrm{Col}(A)$ & by definition of $\mathrm{proj}_W b$, we have that $b-p$ is orthogonal to all $w \in W$. So by Question (2), p is a best approx to b in W, i.e. $\underbrace{\|b-p\|}_{=\|p-b\|} \leq \underbrace{\|b-p'\|}_{=\|p'-b\|}$ $\forall p' \in W$

now $p = A\hat{x}$ & $\forall x \in \mathbb{R}^n$, $Ax = p' \in W$.

so $\forall x \in \mathbb{R}^n$ $\|p-b\| = \|A\hat{x}-b\| \leq \|p'-b\| = \|Ax-b\|$. QED

(b) By part (c) of Question (2), $p = \mathrm{proj}_W b$ is the <u>unique</u> best approximation to b in W. So the only $x \in \mathbb{R}^n$ for which Ax is a ~~best approx. to~~ best approx. to b in W are the solutions of $Ax = p$ (& all such x are best approx.). QED

(c) By part (b) of Question (1), If $A^T v = A^T A x$, then $A x = p$, so x is a least squares solution to $Ax = b$. Conversely, if x is a least squares solution to $Ax = b$, then x is a solution to $Ax = p$, by part (b)

so $A^T v - A^T A x = A^T v - A^T p = A^T(v-p) = 0$

because $v-p$ is orthogonal to the columns of A. QED

(4) $b = \begin{pmatrix} 0 \\ 102 \\ 400 \end{pmatrix}$, $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 4 & 1 \end{pmatrix}$. The least squares sol'ns of $Ax=b$ are the solutions of $A^T A x = A^T b$

(it looks like it should be about $y(t) = 100t$) i.e. $\alpha \approx 100$, $\beta \approx 0$

$$= \begin{pmatrix} 0 & 1 & 4 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 4 & 1 \end{pmatrix} x = \begin{pmatrix} 0 & 1 & 4 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 102 \\ 400 \end{pmatrix}$$

$$= \begin{pmatrix} 17 & 5 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1702 \\ 502 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 17 & 5 & 1702 \\ 5 & 3 & 502 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & \frac{5}{17} & 100+\frac{2}{17} \\ 1 & 3/5 & 100+\frac{2}{5} \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & \frac{5}{17} & 100+\frac{2}{17} \\ 0 & \frac{26}{85} & \frac{24}{85} \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cc|c} 1 & 0 & 100-\frac{2}{13} \\ 0 & 1 & \frac{12}{13} \end{array}\right)$$

$\frac{3}{5} - \frac{5}{17} = \frac{51-25}{85} = \frac{26}{85}$

$\frac{2}{5} - \frac{2}{17} = \frac{34-10}{85} = \frac{24}{85}$

$\frac{24}{26} = \frac{12}{13}$

$\frac{2}{17} - \frac{5}{17} \cdot \frac{12}{13} = \frac{26-60}{17 \cdot 13} = \frac{-34}{17 \cdot 13} = \frac{-2}{13}$

so $\boxed{\alpha = 100 - \frac{2}{13}} \approx 100$

$\boxed{\beta = \frac{12}{13}} \approx 0$

(5) (a) $\bar{U}^T U = I$ & $\bar{A}^T A = I$, then $\overline{(AU)}^T AU = (\bar{A}\bar{U})^T AU = \bar{U}^T \underbrace{\bar{A}^T A}_{I} U = \bar{U}^T U = I$ ✓

(b) $A = \bar{A}^T$ implies $a_{ii} = \bar{a}_{ii}$, so diagonal entries must be real.

let $A = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1+i \\ 1-i & 0 \end{pmatrix}$ both Hermitian, but $(1,1)$ entry of AB is

$i(1-i) = i+1 \notin \mathbb{R}$. ✓

(A general 2x2 Hermitian matrix will look like $\begin{pmatrix} a & b+ci \\ b-ci & d \end{pmatrix}$, a,b,c,d real.

$$\begin{pmatrix} a & b+ci \\ b-ci & d \end{pmatrix}\begin{pmatrix} a' & b'+c'i \\ b'-c'i & d' \end{pmatrix} = \begin{pmatrix} aa' + bb' + cc' + i(b'c - bc') & * \\ * & * \end{pmatrix}$$

so the $(1,1)$-entry is not real whenever $b'c \neq bc'$.

so an even simpler example is $b = 1 = c'$, $b' = c = 0$: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$)

(c) $A = \bar{A}^T$ iff $\overline{a_{ij}} = a_{ji}$. If $A = \bar{A}^T$ & $B = \bar{B}^T$, then

$(\overline{A+B}^T)_{ij} = \overline{a_{ji} + b_{ji}} = \bar{a}_{ji} + \bar{b}_{ji} = a_{ij} + b_{ij} = (A+B)_{ij}$

so $\overline{(A+B)}^T = A+B$ QED

(d) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, is not unitary since the cols have norm $\sqrt{2} \neq 1$.

(e) look at 2x2 case: $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} + \begin{pmatrix} \cos\theta' & -\sin\theta' \\ \sin\theta' & \cos\theta' \end{pmatrix} = \begin{pmatrix} \cos\theta + \cos\theta' & -(\sin\theta + \sin\theta') \\ \sin\theta + \sin\theta' & \cos\theta + \cos\theta' \end{pmatrix}$

need norm of first column to $= 1$. So

$$(\cos\theta + \cos\theta')^2 + (\sin\theta + \sin\theta')^2 = \cos^2\theta + \cos^2\theta' + 2\cos\theta\cos\theta' + \sin^2\theta + \sin^2\theta' + 2\sin\theta\sin\theta'$$
$$= 1 + 1 + 2\underbrace{(\cos\theta\cos\theta' + \sin\theta\sin\theta')}_{\cos(\theta-\theta')}$$

so $2 + 2\cos(\theta - \theta') = 1$ so $\cos(\theta - \theta') = -\frac{1}{2}$ so $\theta - \theta' = 2\pi/3$ or $4\pi/3$

try $\theta' = 0$, $\theta = \frac{2\pi}{3}$: $\begin{pmatrix} -1/2 & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 1/2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \cos\frac{\pi}{3} & -\sin\pi/3 \\ \sin\pi/3 & \cos\pi/3 \end{pmatrix}$

(6) (a) $R_\theta^T R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \cos\theta\sin\theta \\ -\sin\theta\cos\theta + \sin\theta\cos\theta & \cos^2\theta + \sin^2\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ✓ so it is unitary!

$\bar{R}_\theta = R_\theta\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ so $\bar{R}_\theta^T \bar{R}_\theta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^T R_\theta^T R_\theta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ✓

(6)(b) If $R_\theta = \overline{R}_{\theta'}$, then $\cos\theta = \cos\theta'$ (looking at (1,1)-entries), so $\theta = \theta'$ or $2\pi - \theta'$

now $\sin(2\pi - \theta') = \sin(2\pi)\cos(\theta') + \sin(-\theta')\cos(2\pi)$

$= -\sin\theta'$

since $\sin\theta = \sin\theta'$ ((2,1)-entries)

if $\theta = 2\pi - \theta'$, then $\sin\theta' = -\sin\theta'$ so $\theta' = 0$ or π

if $\theta' = 0$, $\theta = 2\pi \notin [0, 2\pi)$ so $\theta = \theta' = \pi$

so $\theta = \theta'$

but (2,2)-entries imply $\cos\theta = -\cos\theta' = -\cos\theta$

so $\cos\theta = 0$, so $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$

(2,1)-entries imply $\sin\theta = -\sin\theta$ so $\theta = 0$ or π not $\frac{\pi}{2}$ or $\frac{3\pi}{2}$

so $R_\theta \neq \overline{R}_{\theta'}$ $\forall\, \theta, \theta' \in [0, 2\pi)$

If $R_\theta = R_{\theta'}$, $\theta \neq \theta'$, then $\cos\theta = \cos\theta'$ so $\theta = \theta'$ or $2\pi - \theta'$

also $\sin\theta = \sin\theta'$ so $\theta = \theta'$ (as shown above)

but $\theta \neq \theta'$ ✓

Similarly $\overline{R}_\theta = \overline{R}_{\theta'}$ requires $\cos\theta = \cos\theta'$ & $\sin\theta = \sin\theta'$. QED

(c) $\theta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $a^2 + c^2 = 1$ & $b^2 + d^2 = 1$ since $a^2, b^2, c^2, d^2 \geq 0$

This implies $a^2, b^2, c^2, d^2 \leq 1$ so $|a|, |b|, |c|, |d| \leq 1$

$\cos\theta$ & $\sin\theta$ both have range $[-1, 1]$ so any number in that range is of the form $\cos\theta$ or $\sin\theta$, as you wish.

(d) If $a = \pm\cos\theta$, then $a^2 + c^2 = 1$ implies $c^2 = 1 - \cos^2\theta = \sin^2\theta$ so $c = \pm\sin\theta$

& $a^2 + b^2 = 1$ which implies $b^2 = 1 - \cos^2\theta = \sin^2\theta$ so $b = \pm\sin\theta$

(e) If $b = \pm\sin\theta$, then $b^2 + d^2 = 1$, so $d^2 = 1 - \sin^2\theta = \cos^2\theta$ so $d = \pm\cos\theta$

(f) If $\begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ then $\begin{pmatrix} a \\ c \end{pmatrix} \cdot \begin{pmatrix} b \\ d \end{pmatrix} = 0$ implies $ab + cd = 0$

by part (e), $ab + cd$ is $\cos\theta(\pm\sin\theta) + \sin\theta(\pm\cos\theta) = 0$

so $\frac{b}{d} = \frac{-\sin\theta}{\cos\theta}$ so $\begin{pmatrix} b \\ d \end{pmatrix} = \pm\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$

(g) There are exactly two $\theta \in [0, 2\pi)$ s.t. $\cos\theta = a$ unless $a = -1$, so that $\theta = \pi$ for $\theta \in [0, \pi]$, $\cos\theta$ has range $[-1, 1]$, so $\exists\, \theta \in [0, \pi]$ with $\cos\theta = a$. Then $2\pi - \theta$ also gives a. we know $c = \pm\sin\theta$ On $[0, \pi]$, $\sin\theta \geq 0$, so if $c \geq 0$, take θ, otherwise take $2\pi - \theta$. Done since $\sin(2\pi - \theta) = -\sin\theta$

(6) (h) $\overline{R}_\theta = R_\theta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$ ie. reflection across the $\underline{\underline{x}}$-axis (oops)

(7) (a) $S_2 = \{id, (12)\}$. $id(P_2) = P_2$ so $\boxed{sgn(id) = 1}$

$P_2(x_1, x_2) = (x_1 - x_2)$ $\quad ((12)P_2)(x_1, x_2) = (x_2 - x_1) = -(x_1 - x_2) = -P_2$

so $\boxed{sgn((12)) = -1}$

(b) $S_3 = \{id, (12), (13), (23), (123), (132)\}$

$P_3(x_1, x_2, x_3) = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$

$id\, P_3 = P_3$ so $\boxed{sgn(id) = 1}$

$(12)P_3 = (x_2 - x_1)(x_2 - x_3)(x_1 - x_3) = -(x_1 - x_2)(x_1 - x_3)(x_2 - x_3) = -P_3$

so $\boxed{sgn((12)) = -1}$

$(13)P_3 = (x_3 - x_2)(x_3 - x_1)(x_2 - x_1) = (-1)^3 P_3$ so $\boxed{sgn((13)) = -1}$

$(23)P_3 = (x_1 - x_3)(x_1 - x_2)(x_3 - x_2) = -P_3$ so $\boxed{sgn((23)) = -1}$

$(123)P_3 = (x_2 - x_3)(x_2 - x_1)(x_3 - x_1) = (-1)^2 P_3$ so $\boxed{sgn((123)) = 1}$

$(132)(P_3) = (x_3 - x_1)(x_3 - x_2)(x_1 - x_2) = (-1)^2 P_3$ $\boxed{\text{so } sgn((132)) = 1}$

(8) (a) $F = \mathbb{R}$: $A = (a_{ij})$, $A^T = -A$ implies $a_{ii} = -a_{ii}$ so $a_{ii} = 0$

$F = \mathbb{F}_2$: get $a_{ii} = -a_{ii}$ but $-1 = 1$, so $a_{ii} = a_{ii}$ is no ~~condition~~ condition.

in fact, $A^T = -A = A$ so A is merely symmetric

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in M_2(\mathbb{F}_2)$ is thus skew-symmetric.

(b) $(\Rightarrow)$ Say $B_A(v, v) = 0$ $\forall v \in V$, ~~Recall~~ ~~for~~ ~~$v = e_i$~~ ~~std basis vector~~

Recall that $B_A(e_i, e_j) = e_i^T A e_j = a_{ij}$ (for $A = (a_{ij})$)

so $B_A(e_i, e_i) = 0 \Rightarrow a_{ii} = 0$ $\forall i$.

we saw in class that $B_A(v, v) = 0$ implies $B_A(v, w) = -B_A(w, v)$

so $a_{ji} = B_A(e_j, e_i) = -B_A(e_i, e_j) = -a_{ij}$ so A is alternating

$(\Leftarrow)$ if A is alternating, then $B_A(e_i, e_j) = -B_A(e_j, e_i)$ & $B_A(e_i, e_i) = 0$

Now $\forall v \in V = F^n$, $\exists \alpha_i \in F$ s.t. $v = \sum_{i=1}^n \alpha_i e_i$. Then

$B_A(v, v) = B_A\left(\sum_{i=1}^n \alpha_i e_i, \sum_{j=1}^n \alpha_j e_j\right) = \sum_{i=1}^n \alpha_i B_A\left(e_i, \sum_{j=1}^n \alpha_j e_j\right) = \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j B_A(e_i, e_j)$

(b) (cont'd)

now $B_A(e_i,e_i)=0$

so $B_A(v,v)=\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\alpha_i\alpha_j B_A(e_i,e_j)$

$=\sum_{i=1}^{n}\left(\sum_{1\le j<i}\alpha_i\alpha_j B_A(e_i,e_j)+\sum_{i<j\le n}\alpha_i\alpha_j B(e_i,e_j)\right)$

$=\sum_{i=1}^{n}\left(\sum_{1\le j<i}\alpha_i\alpha_j B_A(e_i,e_j)+\underbrace{\sum_{i<j\le n}\alpha_i\alpha_j(-B(e_j,e_i))}_{\sum_{1\le j<i}\alpha_i\alpha_j(-B(e_i,e_j))}\right)$

$=\sum_{i=1}^{n}\sum_{1\le j<i}\alpha_i\alpha_j\left(B_A(e_i,e_j)-B_A(e_i,e_j)\right)=\boxed{0}$ QED

(c) If $\begin{pmatrix}a&b\\c&d\end{pmatrix}$ is alternating, then $a=d=0$ & $b=-c$, so $\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}0&b\\-b&0\end{pmatrix}=b\begin{pmatrix}0&1\\-1&0\end{pmatrix}$

so $\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ is a basis of the space of 2x2 alternating matrices over F

As there's a bijection $A\mapsto B_A$ between alt. matrices & alt. bilinear forms, & $B_{A+A'}=B_A+B_{A'}$ & $B_{\lambda A}=\lambda B_A$, the map $A\to B_A$ is an isom. of vector spaces, so the space of alt. bilinear forms is one-dim'l.

If $A=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$, $B_A(v,w)=v^T\begin{pmatrix}0&1\\-1&0\end{pmatrix}w$. This is a basis.

(Note: $(a\ b)\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}c\\d\end{pmatrix}=(-b\ a)\begin{pmatrix}c\\d\end{pmatrix}=-bc+ad$, the determinant of $\begin{pmatrix}a&b\\c&d\end{pmatrix}$!)