Math 411

Solutions to Asst 12

1. • $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$ so $\det = 1\cdot 1\cdot(-2) = \boxed{-2}$

• $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 2 & 3 \\ 3 & 2 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -4 & -5 \\ 0 & -4 & -8 & -10 \\ 0 & -5 & -10 & -15 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -4 & -5 \\ 0 & 0 & -8/3 & -10/3 \\ 0 & 0 & -\frac{10}{3} & -\frac{20}{3} \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -4 & -5 \\ 0 & 0 & -8/3 & -\frac{10}{3} \\ 0 & 0 & 0 & -5/2 \end{pmatrix}$

so $\det = 1\cdot(-3)\cdot\left(-\frac{8}{3}\right)\cdot\left(\frac{-5}{2}\right)$
$= \boxed{-20}$

$0 \ 4 \ \frac{16}{3} \ \frac{20}{3}$; $0 \ 5 \ \frac{20}{3} \ \frac{25}{3}$; $0 \ 0 \ \frac{10}{3} \ \frac{100}{24}$

$0 \ -4 \ -\frac{24}{3} \ -\frac{30}{3}$; $0 \ -5 \ -\frac{30}{3} \ -\frac{45}{3}$; $0 \ 0 \ -\frac{10}{3} \ -\frac{160}{24}$

$\frac{-60}{24} = -\frac{10}{4} = \frac{-5}{2}$

• $\begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix} = -a\begin{vmatrix} -a & c \\ -b & 0 \end{vmatrix} + b\begin{vmatrix} -a & 0 \\ -b & -c \end{vmatrix} = -a\cdot(-(-b)\cdot c) + b\cdot((-a)\cdot(-c))$
$= -abc + abc = \boxed{0}$

• $\begin{vmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{vmatrix} = -a\begin{vmatrix} -a & d & e \\ -b & 0 & f \\ -c & -f & 0 \end{vmatrix} + b\begin{vmatrix} -a & 0 & e \\ -b & -d & f \\ -c & -e & 0 \end{vmatrix} - c\begin{vmatrix} -a & 0 & d \\ -b & -d & 0 \\ -c & -e & -f \end{vmatrix}$

$= -a\begin{vmatrix} -a & d & e \\ -b & 0 & f \\ -c & -f & 0 \end{vmatrix} - b\begin{vmatrix} -b & -d & f \\ -a & 0 & e \\ -c & -e & 0 \end{vmatrix} - c\begin{vmatrix} -c & -e & -f \\ -a & 0 & d \\ -b & -d & 0 \end{vmatrix}$

look like $\begin{vmatrix} m & n & p \\ q & 0 & s \\ r & -s & 0 \end{vmatrix} = r\begin{vmatrix} n & p \\ 0 & s \end{vmatrix} + s\begin{vmatrix} m & p \\ q & s \end{vmatrix}$
$= rns + s^2m - spq$

so $= -a\left((-c)\cdot d\cdot f + f^2\cdot(-a) - fe(-b)\right) - b\left((-c)\cdot(-d)\cdot e + e^2(-b) - e\cdot f(-a)\right)$
$- c\left((-b)\cdot(-e)\cdot d + d^2\cdot(-c) - d(-f)(-a)\right)$
$= \boxed{acdf + a^2f^2 - abef - bcde + b^2e^2 - abef - bcde + c^2d^2 + acdf}$
$\left(= (af - be + cd)^2\right)$

(remark: in fact, the det of a nxn alternating matrix is always 0 when n is odd & is always a square when n is even; it is the square of the "pfaffion" of the matrix.)

(2) $$\begin{vmatrix}1 & x_1 & x_1^2\\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2\end{vmatrix} = \begin{vmatrix}1 & x_1 & x_1^2\\ 0 & x_2-x_1 & x_2^2-x_1^2\\ 0 & x_3-x_1 & x_3^2-x_1^2\end{vmatrix} = (x_2-x_1)(x_3-x_1)\begin{vmatrix}1&x_1&x_1^2\\0&1&x_2+x_1\\0&1&x_3+x_1\end{vmatrix}$$

$$= (x_2-x_1)(x_3-x_1)\begin{vmatrix}1&x_1&x_1^2\\0&1&x_2+x_1\\0&0&x_3-x_2\end{vmatrix} = \boxed{(x_2-x_1)(x_3-x_2)(x_3-x_1)}$$

(3) (a) $A^{-1}A = I$ so $\det(A^{-1}A) = \det(I) = 1$

$\det(A^{-1}A) = \det(A^{-1})\det(A)$ so $\det(A^{-1}) = \dfrac{1}{\det(A)} = (\det A)^{-1}$

(b) $OO^T = I$ & $\det(A) = \det(A^T)$ for all A

so $\det(OO^T) = \det(I) = 1$

$\det(OO^T) = \det(O)\det(O^T) = \det(O)^2$ so $\boxed{\det(O) = \pm 1}$

(c) Note that $\det(\bar{A}) = \overline{\det(A)}$ (for instance, since $\overline{(a_{ij})} = (\overline{a_{ij}})$

& $\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}$

we get $\det(\bar{A}) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} \overline{a_{i\sigma(i)}}$

$= \overline{\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}} = \overline{\det(A)}$)

so $\det(A^*) = \det(\bar{A}^T) = \overline{\det(A)}$

so if $UU^* = 1$, then $\det(U)\overline{\det(U)} = 1$ i.e. $\boxed{|\det(U)| = 1}$ QED

(d) $\det(A) = \det(A^*) = \overline{\det(A)}$ so $\det(A)$ is real. QED

(4) $$\det A = \begin{vmatrix}1&2&3\\4&5&6\\7&8&10\end{vmatrix} = \begin{vmatrix}1&2&3\\0&-3&-6\\0&-6&-11\end{vmatrix} = \begin{vmatrix}1&2&3\\0&-3&-6\\0&0&1\end{vmatrix} = -3$$

$0 \quad 5 \quad 10$
$0 \quad -5 \quad -11$

$$\begin{vmatrix}1&2&3\\1&5&6\\2&8&10\end{vmatrix} = \begin{vmatrix}1&2&3\\0&3&3\\0&4&4\end{vmatrix} = 0 \qquad \begin{vmatrix}1&1&3\\4&1&6\\7&2&10\end{vmatrix} = \begin{vmatrix}1&1&3\\0&-3&-6\\0&-5&-11\end{vmatrix} = \begin{vmatrix}1&1&3\\0&-3&-6\\0&0&-1\end{vmatrix} = 3$$

$$\begin{vmatrix}1&2&1\\4&5&1\\7&8&2\end{vmatrix} = \begin{vmatrix}1&2&1\\0&-3&-3\\0&-6&-5\end{vmatrix} = \begin{vmatrix}1&2&1\\0&-3&-3\\0&0&1\end{vmatrix} = -3$$ so sol'n is $\begin{pmatrix}0/3\\-3/3\\+3/3\end{pmatrix} = \boxed{\begin{pmatrix}0\\-1\\+1\end{pmatrix}}$

(5) (a) ~~To move the~~ row expansion along first row gives

$$\det = (-1)^{n+1} c_0 \overbrace{\begin{vmatrix} 1 & 0 & & 0 \\ 0 & 1 & & \\ & & \ddots & 0 \\ 0 & & 0 & 1 \end{vmatrix}}^{(n-1)\times(n-1)\text{ id. matrix}} = (-1)^{n+1} c_0 \quad \text{QED}$$

(b) for $n=1$, $\det(xI-A) = \det((x)) = x$ ✓

assume it's true for $(n-1)\times(n-1)$ matrix, i.e.

$$\begin{vmatrix} x & & & & -c_1 \\ -1 & x & & & -c_2 \\ & -1 & x & & \vdots \\ & & -1 & \ddots & \vdots \\ & & & \ddots & \vdots \\ & & & -1 & x-c_{n-1} \end{vmatrix} = x^{n-1} - c_{n-1}x^{n-2} - \cdots - c_2 x - c_1$$

then $\det(xI-A) =$

$$\begin{vmatrix} x & 0 & 0 & \cdots & -c_0 \\ -1 & x & 0 & & -c_1 \\ 0 & -1 & x & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & & -1 & x-c_{n-1} \end{vmatrix} = x \underbrace{\begin{vmatrix} x & 0 & & -c_1 \\ -1 & x & & -c_2 \\ & -1 & \ddots & \vdots \\ & & \ddots & \vdots \\ & & -1 & x-c_{n-1} \end{vmatrix}}_{=\,x^{n-1} - c_{n-1}x^{n-2} - \cdots - c_2 x - c_1} + (-1)^{n+1}(-c_0) \cdot \underbrace{\begin{vmatrix} -1 & x & & & \\ 0 & -1 & x & & 0 \\ & 0 & -1 & \ddots & \\ & & & \ddots & 0 \\ & 0 & & & 0 \\ & & & 0 & -1 \end{vmatrix}}_{(-1)^{n-1}}$$

so $= x(x^{n-1} - c_{n-1}x^{n-2} - \cdots - c_2 x - c_1) + (-c_0)$

$= x^n - c_{n-1}x^{n-1} - \cdots - c_2 x^2 - c_1 x - c_0$ QED