Math 411
Solutions to Asst 13

(1)(a) if $\alpha_1 = 0$, then $\alpha_1 I - A = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & \alpha_2\alpha_3 \\ 0 & -1 & -\alpha_2-\alpha_3 \end{pmatrix}$ which has 2 pivots independent of the values of α_2 & α_3, so $\dim \ker(\alpha_1 I - A) = 1$.

Since $\alpha_1, \alpha_2, \alpha_3$ are arbitrarily labelled, the same is true if you assume $\alpha_2 = 0$ or $\alpha_3 = 0$. QED

(b) let $\alpha_i, \alpha_j, \alpha_k$ be the three roots

Then $\alpha_i I - A = \begin{pmatrix} \alpha_i & 0 & -\alpha_i\alpha_j\alpha_k \\ -1 & \alpha_i & \alpha_i\alpha_j + \alpha_i\alpha_k + \alpha_j\alpha_k \\ 0 & -1 & -\alpha_j - \alpha_k \end{pmatrix} \rightsquigarrow \begin{pmatrix} -1 & \alpha_i & \alpha_i\alpha_j + \alpha_i\alpha_k + \alpha_j\alpha_k \\ 0 & -1 & -\alpha_j - \alpha_k \\ 0 & \alpha_i^2 & \alpha_i^2\alpha_j + \alpha_i^2\alpha_k \end{pmatrix}$

$-\alpha_i R_2 = \alpha_i \quad -\alpha_i^2 \quad -\alpha_i^2\alpha_j - \alpha_i^2\alpha_k - \alpha_j\alpha_j\alpha_k$

$R_1 = \alpha_i \quad 0 \quad -\alpha_i\alpha_j\alpha_k$

$\downarrow$

$\begin{pmatrix} -1 & \alpha_i & \alpha_i\alpha_j + \alpha_i\alpha_k + \alpha_j\alpha_k \\ 0 & -1 & -\alpha_j - \alpha_k \\ 0 & 1 & \alpha_j + \alpha_k \end{pmatrix}$

$\rightsquigarrow \begin{pmatrix} -1 & \alpha_i & \alpha_i\alpha_j + \alpha_i\alpha_k + \alpha_j\alpha_k \\ 0 & -1 & -\alpha_j - \alpha_k \\ 0 & 0 & 0 \end{pmatrix}$

so dim ker $= 1$

(2)(a) $xI - A$ is diagonal, $n \times n$, with diagonal entries $x - \lambda$, so $\det = (x-\lambda)^n$. QED

(b) $\lambda I - A = \begin{pmatrix} 0 & -1 & & & 0 \\ 0 & 0 & -1 & & \\ 0 & 0 & 0 & -1 & \\ & & & \ddots & -1 \\ & & & & 0 \end{pmatrix}$ so $n-1$ pivots so dim ker $= 1$ QED

(3) <u>Claim 1:</u> If $A = \left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & I_{n_2} \end{array}\right)$, where I_{n_2} is $n_2 \times n_2$ identity matrix, then $\det(A) = \det(A_1)$

~~proof: induction on n_2: if $n_2 = 1$: $\det(A) = $~~ $\cancel{\left|\begin{array}{c|c} A & 0 \\ \hline 0 & 1 \end{array}\right|}$ ~~row expansion along~~

& If $A = \left(\begin{array}{c|c} I_{n_1} & 0 \\ \hline 0 & A_2 \end{array}\right)$ then $\det A = \det A_2$.

proof: first: $A = \left(\begin{array}{c|c} I_{n_1} & 0 \\ \hline 0 & A_2 \end{array}\right)$: induction on n_1: if $n_1 = 1$, $\det(A) = \left|\begin{array}{c|c} 1 & \\ \hline & A_2 \end{array}\right| = 1 \cdot \det(A_2)$ ✓ row exp'n along 1st row

for $n_1 > 1$: view $A = \left(\begin{array}{c|c} I_{n_1} & 0 \\ \hline 0 & A_2 \end{array}\right)$ as $= \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & A_2' \end{array}\right)$

where $A_2' = \left(\begin{array}{c|c} I_{n_1-1} & 0 \\ \hline 0 & A_2 \end{array}\right)$. Then as in $n_1 = 1$ case, $\det(A) = \det(A_2')$ & by ind. hypothesis $\det(A_2') = \det(A_2)$

(3) for $A = \left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & I_{n_2} \end{array}\right)$, do the same thing. QED

Claim 2: suppose $A = \left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right)$, then $\det(A) = \det(A_1)\det(A_2)$

proof: note $A = \underbrace{\left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & I_{n_2} \end{array}\right)}_{A'} \underbrace{\left(\begin{array}{c|c} I_{n_1} & 0 \\ \hline 0 & A_2 \end{array}\right)}_{A''}$

Then $\det(A) = \det(A')\det(A'') = \underbrace{\det(A_1)\det(A_2)}_{\text{by Claim 1}}$ QED

Claim 3: if $A = \bigoplus_{i=1}^{k} A_i$, then $\det(A) = \prod_{i=1}^{k} \det(A_i)$

proof: induction on k: $k=2$: is claim 2

suppose true for $k-1$ blocks. $A = A_1 \oplus A_2'$ where $A_2' = \bigoplus_{i=2}^{k} A_i$

by claim 2, $\det(A) = \det(A_1)\det(A_2')$

by ind. hypothesis, $\det(A_2') = \prod_{i=2}^{k} \det(A_i)$ QED.

(4)(a) let $W = \mathbb{R}^{n+1}$ & let $\varphi: V \xrightarrow{\sim} W$ given by $\varphi(x^i) = e_{i+1}$, i.e. the "coordinates with respect to $B = \{1, x, x^2, \ldots x^n\}$" isomorphism. $\frac{d}{dx} x^i = i x^{i-1}$ so

Then $\left[\frac{d}{dx}\right]_B = \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 2 & & 0 & \\ & & 0 & 3 & & \\ & & & 0 & \ddots & \\ & 0 & & & \ddots & n \\ & & & & & 0 \end{pmatrix}$ so $\det\left(\frac{d}{dx}\right) = \det\left(\left[\frac{d}{dx}\right]_B\right) = \boxed{0}$

(c) $\mathrm{char}_{\frac{d}{dx}}(t) = \det\left(t \cdot I - \left[\frac{d}{dx}\right]_B\right) = \begin{vmatrix} t & -1 & & \\ & t & -2 & 0 \\ & & \ddots & \ddots & \\ & 0 & & \ddots & -n \\ & & & & t \end{vmatrix} = t^{n+1}$

(b) by (c), $\lambda = 0$ is the only e-value. Then $\underbrace{\mathrm{Ker}\det\left(\frac{d}{dx} - 0\right)}_{} = \varphi^{-1}\left(\mathrm{Null}\left(\left[\frac{d}{dx}\right]_B - 0\right)\right)$

& $\mathrm{Null}\left(\left[\frac{d}{dx}\right]_B\right)$ is 1-dim'l with basis $\left\{\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}\right\}$. so $E_0 = \varphi^{-1}\left(\mathrm{Span}\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}\right)$ = constants

QED

(5) (a) $A_1 - 2 = \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix} = 0$, $A_2 - 2 = \begin{pmatrix} 0&0&0\\0&0&1\\0&0&0 \end{pmatrix} \neq 0$, $A_3 - 2 = \begin{pmatrix} 0&1&0\\0&0&1\\0&0&0 \end{pmatrix} \neq 0$

$(A_2-2)^2 = \begin{pmatrix} 0&0&0\\0&0&1\\0&0&0 \end{pmatrix}\begin{pmatrix} 0&0&0\\0&0&1\\0&0&0 \end{pmatrix} = \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix} = 0$ $(A_3-2)^2 = \begin{pmatrix} 0&1&0\\0&0&1\\0&0&0 \end{pmatrix}\begin{pmatrix} 0&1&0\\0&0&1\\0&0&0 \end{pmatrix} = \begin{pmatrix} 0&0&1\\0&0&0\\0&0&0 \end{pmatrix} \neq 0$

$(A_3-2)^3 = \begin{pmatrix} 0&1&0\\0&0&1\\0&0&0 \end{pmatrix}\begin{pmatrix} 0&0&1\\0&0&0\\0&0&0 \end{pmatrix} = \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix} = 0$ QED

b) From we see that there are 0, 1, & 2 pivots so dim Ker is 3, 2, 1, respectively.

c) A is diag'ble iff there's a basis of e-vectors. since $\ker(A_1 - 2) = F^3$, there's a basis of F^3 consisting of e-vectors of A_1, so A_1 is diag'ble.

$\mathrm{Char}_{A_2}(x) = \begin{vmatrix} x-2&0&0\\0&x-2&-1\\0&0&x-2 \end{vmatrix} = (x-2)^3$ & $\mathrm{Char}_{A_3}(x) = \begin{vmatrix} x-2&-1&0\\0&x-2&-1\\0&0&x-2 \end{vmatrix} = (x-2)^3$

so their only e-value is $\lambda = 2$. since $\ker(A_i - 2) \subsetneq F^3$ for $i = 2,3$, there is no basis of F^3 of e-vectors of A_i, $i = 2,3$, so A_i is not diag'ble for $i = 2,3$ QED

(d) $\mathrm{Char}_{A_1}(x) = \begin{vmatrix} x-2&0&0\\0&x-2&0\\0&0&x-2 \end{vmatrix} = (x-2)^3$. Char_{A_2} & Char_{A_3} were computed above.

(6) (a) Part (a) of (5) says that min. poly of A_i is $f_i(x)$.

(b) Since min poly divides char poly $= x^{n+1}$ & x divides the min poly, we just need to show that $\left(\frac{d}{dx}\right)^j \neq 0$ $\forall\, 0 \le j \le n$. Well, let $v = x^n$. Then $\left(\frac{d}{dx}\right)^j(v) = n(n-1)\cdots(n-(j-1))\,x^{n-j} \neq 0$

QED

T is diag'ble iff its min poly is a product of distinct linear factors. if $n \geq 1$, then x^{n+1} is not such, so T is <u>not</u> diag'ble. QED

(7) (a) one could find a matrix for T, but in fact note that $T(x^j) = x\frac{d}{dx}x^j = jx^j$ so $v_j = x^j$ is an e-vector with e-value $\boxed{\lambda_j = j \text{ for } j = 0, \ldots, n}$. Since an operator on an $(n+1)$-dim'l space has at most $(n+1)$-evalues, & $\lambda_j = j$, $j = 0, \ldots n$ is $(n+1)$ (distinct) e-values, these are all the e-values of T

(b) From (a), $\mathrm{Char}_T(x) = \prod_{j=0}^{n}(x-j)$ (c) since $m_T(x)$ divides $\mathrm{char}_T(x)$ & the latter is a product of distinct factors, $m_T(x) = \mathrm{char}_T(x)$. (d) since $m_T(x)$ is a prod of distinct linear factors, T is diag'ble.