Math 411

Asst 14 Solutions

(1) $\lambda_1 = 2$, $\lambda_2 = 3$. The char. poly tells us that: dim of gen'd e-space of 2 is $\boxed{2}$ & ― ― ― ― 3 is $\boxed{2}$

A: $(A-2) = \begin{pmatrix} 1 & 10 & -3 & 10 \\ 1 & 3 & -1 & 3 \\ 7 & -35 & 8 & -28 \\ 1 & -12 & 3 & -10 \end{pmatrix} \to \begin{pmatrix} 1 & 10 & -3 & 10 \\ 0 & -7 & 2 & -7 \\ 0 & -105 & 29 & -98 \\ 0 & -22 & 6 & -20 \end{pmatrix} \to \begin{pmatrix} 1 & 10 & -3 & 10 \\ 0 & 1 & -\frac{2}{7} & 1 \\ 0 & 0 & -1 & 7 \\ 0 & -1 & 0 & 1 \end{pmatrix}$

$\to \begin{pmatrix} 1 & 10 & -3 & 10 \\ 0 & 1 & -\frac{2}{7} & 1 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & -\frac{2}{7} & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 10 & -3 & 10 \\ 0 & 1 & -\frac{2}{7} & 1 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

so $\boxed{\text{dim of 2-e-space} = 1}$

$1 < 2$ so A is $\boxed{\text{not dgble}}$

~~so min poly~~

$(A-3) = \begin{pmatrix} 0 & 10 & -3 & 10 \\ 1 & 2 & -1 & 3 \\ 7 & -35 & 7 & -28 \\ 1 & -12 & 3 & -11 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 10 & -3 & 10 \\ 0 & -49 & 14 & -49 \\ 0 & -14 & 4 & -14 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -\frac{3}{10} & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -10 & 0 \end{pmatrix}$

$\to \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -\frac{13}{10} & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\boxed{\text{so dim of 3-e-space} = 1}$

so min poly is strictly "bigger" than $(x-2)(x-3)$ & "less than" $(x-2)^2(x-3)^2$ & so is $\boxed{(x-2)^2(x-3)^2}$. This implies that the Jordan form is

$$\boxed{\left(\begin{array}{cc|cc} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ \hline 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{array}\right)}$$

(1) B: $(B-2) = \begin{pmatrix} 1 & 3 & -1 & 3 \\ 1 & -4 & 1 & -4 \\ 7 & -35 & 8 & -28 \\ 1 & -5 & 1 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & -1 & 3 \\ 0 & -7 & 2 & -7 \\ 0 & -56 & 15 & -49 \\ 0 & -8 & 2 & -6 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & -1 & 3 \\ 0 & -7 & 2 & -7 \\ 0 & 0 & -1 & 7 \\ 0 & 1 & 0 & -1 \end{pmatrix}$

$\to \begin{pmatrix} 1 & 3 & -1 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 7 \\ 0 & 0 & 2 & -14 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & -1 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

so dim of 2-espace $= \boxed{1}$

$1<2$ so B is $\boxed{\text{not diag'ble}}$

$(B-3) = \begin{vmatrix} 0 & 3 & -1 & 3 \\ 1 & -5 & 1 & -4 \\ 7 & -35 & 7 & -28 \\ 1 & -5 & 1 & -4 \end{vmatrix} \to \begin{pmatrix} 1 & -5 & 1 & 4 \\ 0 & 3 & -1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ so dim of 3-espace $= \boxed{2}$

since dim of 3-espace = dim of gen'd e-space of 3, the Jordan block for 3 is diagonal

& ~~the~~ min poly is $\boxed{(x-2)^2(x-3)}$ & Jordan form is $\boxed{\left(\begin{array}{cc|cc} 2 & 1 & & \\ 0 & 2 & & 0 \\ \hline & & 3 & 0 \\ 0 & & 0 & 3 \end{array}\right)}$

(2) (a) $A = PJP^{-1}$: induction on n: $n=1$: clear.

induction hypothesis: suppose $A^{n-1} = PJ^{n-1}P^{-1}$

Then $A^n = APJ^{n-1}P^{-1} = PJ\cancel{P^{-1}P}J^{n-1}P^{-1} = PJ^nP^{-1}$ (QED)

(b) $\text{Char}_A(x) = \begin{vmatrix} x-1 & -1 \\ -1 & x \end{vmatrix} = (x-1)x-1 = x^2-x-1$ so $\lambda_+ = \frac{1+\sqrt{5}}{2}$ the "golden ratio".

& $\lambda_- = \frac{1-\sqrt{5}}{2}$

$\lambda_\pm - A = \begin{pmatrix} -\frac{1}{2} \pm \frac{\sqrt{5}}{2} & -1 \\ -1 & \frac{1}{2} \pm \frac{\sqrt{5}}{2} \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -(\lambda_\pm) \\ 0 & 0 \end{pmatrix}$ so there must be a 0 row.

so let $v_\pm = \begin{pmatrix} \lambda_\pm \\ 1 \end{pmatrix}$, then $Av_\pm = \lambda_\pm v_\pm$, let $P = (v_+ \; v_-) = \boxed{\begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix}}$

Then $P^{-1}AP = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix} = \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix} = J$

(2)(c) Claim: $v_{n+1} = A^n v_1$

proof: induction: $n=1$: $v_2 = Av_1$ yes.

say $v_n = A^{n-1} v_1$.

since $v_{n+1} = A v_n$

get $v_{n+1} = A A^{n-1} v_1 = A^n v_1$ QED

Since $A = PJP^{-1}$ $\quad J = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}$ $\quad \det(P) = \lambda_+ - \lambda_- = \sqrt{5}$

$A^n = P J^n P^{-1}$. Now, $J^n = \begin{pmatrix} \lambda_+^n & 0 \\ 0 & \lambda_-^n \end{pmatrix}$. $\quad P^{-1} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -\lambda_- \\ -1 & \lambda_+ \end{pmatrix}$

$$\text{so } A^n = \begin{pmatrix} \lambda_+ & \lambda_- \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_+^n & 0 \\ 0 & \lambda_-^n \end{pmatrix} \begin{pmatrix} 1 & -\lambda_- \\ -1 & \lambda_+ \end{pmatrix} \frac{1}{\sqrt{5}}$$

$$= \begin{pmatrix} \lambda_+^{n+1} & \lambda_-^{n+1} \\ \lambda_+^n & \lambda_-^n \end{pmatrix} \begin{pmatrix} 1 & -\lambda_- \\ -1 & \lambda_+ \end{pmatrix} \frac{1}{\sqrt{5}} = \frac{1}{\sqrt{5}} \begin{pmatrix} \lambda_+^{n+1} - \lambda_-^{n+1} & \lambda_+ \lambda_-^{n+1} - \lambda_- \lambda_+^{n+1} \\ \lambda_+^n - \lambda_-^n & \lambda_+ \lambda_-^n - \lambda_- \lambda_+^n \end{pmatrix}$$

$$\text{so } \begin{pmatrix} F_{n+2} \\ F_{n+1} \end{pmatrix} = A^n v_1 = \begin{pmatrix} & \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} \cdots \\ \lambda_+^n - \lambda_-^n + \lambda_+ \lambda_-^n - \lambda_- \lambda_+^n \end{pmatrix}$$

$$\text{so } F_{n+1} = \frac{\lambda_+^n (1 - \lambda_-) - \lambda_-^n (1 - \lambda_+)}{\sqrt{5}}$$ note that $1 - \lambda_\pm = \lambda_\mp$

$$\text{so } F_{n+1} = \frac{\lambda_+^{n+1} - \lambda_-^{n+1}}{\sqrt{5}} \qquad \text{so } \boxed{F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}}$$

(also, since $\lambda_+^2 = \lambda_+ + 1$

get $\lambda_+ = 1 + \frac{1}{\lambda_+}$ so $1 - \lambda_- = 1 + \frac{1}{\lambda_+}$ so $\lambda_- = \frac{-1}{\lambda_+}$

$$\text{so } F_n = \frac{\lambda_+^n - \left(-\frac{1}{\lambda_+}\right)^n}{\sqrt{5}} \text{ also)}$$

(3) Induction: $n=1$: clear. say $J^{n-1} = \begin{pmatrix} \lambda^{n-1} & (n-1)\lambda^{n-2} \\ 0 & \lambda^{n-1} \end{pmatrix}$, then $J^n = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} \lambda^{n-1} & (n-1)\lambda^{n-2} \\ 0 & \lambda^{n-1} \end{pmatrix}$

$$= \begin{pmatrix} \lambda & (n-1)\lambda^{n-1} + \lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix} = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}$$

QED

(4) let $A=\begin{pmatrix}4&-4\\1&0\end{pmatrix}$ let $v_n=\begin{pmatrix}a_{n+1}\\a_n\end{pmatrix}$, Then $v_{n+1}=Av_n$ so $v_{n+1}=A^n v_1$

let's do like in Question (2): $\mathrm{char}_A(x)=\begin{vmatrix}x-4&4\\-1&x\end{vmatrix}=x^2-4x+4=(x-2)^2$

so $\lambda=2$ is the only e-value.

$A-2$: $\begin{pmatrix}2&-4\\1&-2\end{pmatrix}\rightsquigarrow\begin{pmatrix}1&-2\\0&0\end{pmatrix}$ so A is not diag'ble (since dim ker of $A-2$ is $1<2$)

$(A-2)^2=\begin{pmatrix}2&-4\\1&-2\end{pmatrix}\begin{pmatrix}2&-4\\1&-2\end{pmatrix}=\begin{pmatrix}4-4&-8+8\\2-2&-4+4\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$

let $v_2=\begin{pmatrix}0\\1\end{pmatrix}$, let $v_1=(A-2)v_2=\begin{pmatrix}2&-4\\1&-2\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}-4\\-2\end{pmatrix}$ so $Av_2=2v_2+v_1$

Then $Av_1=2v_1$ & letting $P=(v_1\ v_2)=\begin{pmatrix}-4&0\\-2&1\end{pmatrix}$, get $P^{-1}AP=\begin{pmatrix}2&1\\0&2\end{pmatrix}=J$

$\det(P)=-4$

Then $A^n=PJ^nP^{-1}$. By (3), $J^n=\begin{pmatrix}2^n&n2^{n-1}\\0&2^n\end{pmatrix}$ $\quad P^{-1}=\frac{1}{-4}\begin{pmatrix}1&0\\2&-4\end{pmatrix}=\begin{pmatrix}-\frac14&0\\-\frac12&1\end{pmatrix}$

$\begin{pmatrix}a_{n+2}\\a_{n+1}\end{pmatrix}=v_{n+1}=A^nv_1=-\frac14\begin{pmatrix}-4&0\\-2&1\end{pmatrix}\begin{pmatrix}2^n&2^{n-1}n\\0&2^n\end{pmatrix}\begin{pmatrix}1&0\\2&-4\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}$

$=-\frac14\begin{pmatrix}-2^{n+2}&-2^{n+1}n\\-2^{n+1}&2^n-2^nn\end{pmatrix}\begin{pmatrix}1\\-2\end{pmatrix}=-\frac14\begin{pmatrix}-2^{n+2}+2^{n+2}n\\-2^{n+1}+2^{n+1}n\end{pmatrix}=\begin{pmatrix}2^n(1-n)\\\cdots\end{pmatrix}$

so $a_{n+2}=2^n(1-n)$

so $\boxed{a_n=2^{n-2}(3-n)}$

i.e. $a_n=3\cdot2^{n-2}-n2^{n-2}$

(remark: These linear recurrence relations are like linear diff'l equations. For instance, the "general sol'n" to the above relation, without specifying a_1 & a_2, ie $a_n=C_12^n+C_2n2^n$)

(5) (a) The char. poly of A is of the form $(x-1)^{m_1}(x-2)^{m_2}(x+7)^{m_3}$ with $m_1\geq2$, $m_2\geq1$, $m_3\geq3$ & $m_1+m_2+m_3=8$

so possibilities are $\boxed{(m_1,m_2,m_3)=\overset{(i)}{(2,1,5)},\ \overset{(ii)}{(2,2,4)},\ \overset{(iii)}{(2,3,3)},\ \overset{(iv)}{(3,1,4)},\ \overset{(v)}{(3,2,3)},\ \overset{(vi)}{(4,1,3)}}$

(5)(b) It cannot be diag'ble because its min. poly. is not a product of distinct linear factors

Possible Jordan forms (corresponding to each of the possibilities (i) - (vi) of part (a))

If $m_1=2$, then the 1 block is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

If $m_1=3$, - - - 1 block is $\left(\begin{array}{c|cc} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right)$

and that a 2×2 block must occur

If $m_1=4$, - —— - - - is $\left(\begin{array}{cc|cc} 1 & 0 & & \\ 0 & 1 & & 0 \\ \hline & & 1 & 1 \\ & 0 & 0 & 1 \end{array}\right)$ or $\left(\begin{array}{cc|cc} 1 & 1 & & \\ 0 & 1 & & 0 \\ \hline & & 1 & 1 \\ & 0 & 0 & 1 \end{array}\right)$

(That the min poly has $(x-2)^2$ in it says biggest Jordan block for $\lambda=1$ is 2×2 & m_1 is the size of the contribution of $\lambda=1$)

Since the min poly has only $(x-2)$, the contribution of $\lambda=2$ is diagonal, of size $m_2\times m_2$.

Since $(x+7)^3$ is what divides the min poly, a 3×3 Jordan block must occur & is the biggest that can occur.

$m_3=3$: $\begin{pmatrix} -7 & 1 & \\ & -7 & 1 \\ & & -7 \end{pmatrix}$ $m_3=4$: $\left(\begin{array}{c|ccc} -7 & & & \\ \hline & -7 & 1 & \\ & & -7 & 1 \\ & & & -7 \end{array}\right)$ $m_3=5$: $\left(\begin{array}{cc|ccc} -7 & 0 & & 0 & \\ 0 & -7 & & & \\ \hline & & -7 & 1 & \\ & 0 & & -7 & 1 \\ & & & & -7 \end{array}\right)$

or $\left(\begin{array}{cc|ccc} -7 & 1 & & 0 & \\ & -7 & & & \\ \hline & & -7 & 1 & \\ & 0 & & -7 & 1 \\ & & & & -7 \end{array}\right)$

so # of possibilities: (i) $1\cdot 1\cdot 2=\boxed{2}$

(ii) $1\cdot 1\cdot 1=\boxed{2}$

(iii) $1\cdot 1\cdot 1=\boxed{1}$

(iv) $1\cdot 1\cdot 1=\boxed{1}$

(v) $1\cdot 1\cdot 1=\boxed{1}$

(vi) $2\cdot 1\cdot 1=\boxed{2}$

so $\boxed{\text{9 possibilities, as described above}}$