Math 411

Asst. 2 solutions

(1) (a) Type III: REF is about 0 entries versus non-zero entries. Multiplying a row by a non-zero scalar doesn't change where the zero & non-zero entries are.

Type II: For $i<j$, the operation $r_i \rightsquigarrow r_i + \alpha r_j$: REF is about the entries below & to the left of a leading entry being zero. Since $i<j$, all the entries of row j below & to the left of the leading entry of row i are zero. Thus, adding row j (or a multiple of it) to row i only affects entries to the right of the leading entry; these entries are unrelated to the REF conditions.

(b) Let A be any matrix. $\exists A'$ in REF s.t. $A \sim A'$.

Induction on number of rows of A:

(i) base case: $m=1$ row: If A is the zero matrix, A is already in RREF

If A is non-zero, let a_{1j} be its leftmost non-zero entry.

applying $r_1 \rightsquigarrow \frac{1}{a_{1j}} r_1$ produces an RREF matrix row equivalent to A.

(ii) induction step: say the case where a matrix has $m-1$ rows is known.

~~Say A is... Let a_{ij} be the leading entries of A', $1 \le i \le M \le m$~~

~~apply the row operations a~~

let B' be the bottom $m-1$ rows of A'. By induction hypothesis, $\exists B'' \sim B'$ s.t. B'' is in RREF. Let A'' be the first row of A' with B'' below.

A'' is in REF. Let a_{ij_i} be its leading entries, $1 \le i \le M \le m$.

Apply $r_1 \rightsquigarrow r_1 - a_{1j_2} r_2$ to make the entry above a_{2j_2} zero.

Then apply $r_1 \rightsquigarrow r_1 - a_{1j_3} r_3$ for the entry in the first row above a_{3j_3} to be 0.

Keep going all the way to $r_1 \rightsquigarrow r_1 - a_{1j_M} r_M$. ~~This produces~~

Finally, apply $r_1 \rightsquigarrow \frac{1}{a_{1j_1}} r_1$. This produces a matrix in RREF.

(2) Say the non-zero rows of A are rows 1 through M and those of B are 1 through M'

If $M \geq N$, all the leading entries of $C = (A\ B)$ are the leading entries of A. They are thus in echelon form & all equal to 1. All the zero rows are below the non-zero rows because this is true for A & $M \geq N$. All the entries above the leading entries are 0 since this is true for A.

(2) (cont'd) If $m < N$, some of the leading entries are from B. All of these are to the right & below those of A, so the matrix is still row echelon form. All the leading entries are 1 because they are leading entries of A or B. Again, the entries above the leading entries are 0 because this is true for both A & B.

(3) With 0 non-zero rows: $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ 0.5 each

/2 With 1 non-zero row: $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix}$ $a \in F$ arbitrary

With 2 non-zero rows: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(4) With 0 non-zero rows: $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

/5 With 1 non-zero row: $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

With 2 non-zero rows: $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

(5) let $f(x) = x$, $g(x) = x^2$. $\mathbb{F}_2 = \{0, 1\}$ so $\tilde{f}(0) = 0$, $\tilde{f}(1) = 1$

/2 $\tilde{g}(0) = 0^2 = 0$, $\tilde{g}(1) = 1^2 = 1$ so $\tilde{f} = \tilde{g}$ though $f \neq g$.

(6) Subspace test: let $f, g \in W$. let $n_1, \dots, n_i$ be s.t. $f(n) = 0$ for $n \neq n_1, \dots, n_i$ & let $m_1, \dots, m_j$ be s.t. $g(m) = 0$ for $m \neq m_1, \dots, m_j$

Then, since $(f+g)(n) = f(n) + g(n)$, we have that $(f+g)(n) = 0$ for $n \neq n_1, \dots, n_i, m_1, \dots, m_j$ so $f + g \in W$.

If $\alpha \in F$, then $(\alpha \cdot f)(n) = \alpha f(n)$, so $(\alpha \cdot f)(n) = 0$ for $n \neq n_1, \dots, n_i$ so $\alpha \cdot f \in W$

QED

(7) (a) let $\varphi(0)=0$, $\varphi(1)=1$, $\varphi(n)=\overbrace{1+1+1\cdots+1}^{n\text{ times}}$ for $n>0$

$\quad$ & $\varphi(-n)=-\varphi(n)$ for $n>0$

Then: (i) $\varphi(0)=0$ ✓

(ii) $\varphi(1)=1$ ✓

(iii) $\varphi(m+n)=\overbrace{1+1+1+\cdots+1}^{m+n\text{ times}}=\overbrace{1+1+1\cdots+1}^{m\text{ times}}+\overbrace{1+1+\cdots+1}^{n\text{ times}}=\varphi(m)+\varphi(n)$ if $m,n>0$

~~(iv) $\varphi(mn)=\underbrace{1+1+1\cdots+1}_{mn\text{ times}}$~~ If $mn=0$, clearly $\varphi(m+n)=\varphi(m)+\varphi(n)$.

if $m,n<0$, $\varphi(m+n)=-\varphi((-m)+(-n))\overset{\text{by previous line}}{=}-(\varphi(-m)+\varphi(-n))$

$=-\varphi(-m)+-\varphi(-n)$

$=\varphi(m)+\varphi(n)$

if $m>0, n<0$; if $m+n>0$, $\varphi(m+n)=\underbrace{1+1+\cdots+1}_{m-|n|\text{ times}}=\underbrace{1+1\cdots+1}_{m\text{ times}}-(\underbrace{1+1+\cdots+1}_{|n|\text{ times}})$

$=\varphi(m)-\varphi(-n)$

$=\varphi(m)+\varphi(n)$ ✓

if $m+n<0$, $\varphi(m+n)=-\varphi((-m)+(-n))$

let $m'=-n$ & $n'=-m$ so $m'>0, n'<0, m'+n'>0$.

apply above ~~argument~~ result to show $\varphi(m'+n')=\varphi(m')+\varphi(n')$

so $-\varphi((-m)+(-n))=-\varphi(-n)-\varphi(-m)$

$=\varphi(n)+\varphi(m)$ ✓

if $m<0, n<0$: let $m'=n$ & $n'=m$ & apply above

(iv) If $mn=0$, $\varphi(m)=0$ or $\varphi(n)=0$ so $\varphi(mn)=0=\varphi(m)\varphi(n)$

If $m,n>0$, $\varphi(mn)=\underbrace{1+1+\cdots+1}_{mn}=\underbrace{(\underbrace{1+1+\cdots+1}_{n})+(\underbrace{1+\cdots+1}_{n})+\cdots+(\underbrace{1+1+\cdots+1}_{n})}_{m}$

$=\underbrace{\varphi(n)+\cdots+\varphi(n)}_{m}=(\underbrace{1+1+\cdots+1}_{m})\varphi(n)$

$=\varphi(m)\varphi(n)$

If $m,n<0$, $\varphi(mn)=\varphi((-m)(-n))=\varphi(-m)\varphi(-n)$

$=(-\varphi(-m))(-\varphi(-n))=(-1)^2\varphi(m)\varphi(n)$

$=\varphi(m)\varphi(n)$ ✓

If $m>0, n<0$: $\varphi(mn)=-\varphi(m(-n))=-\varphi(m)\varphi(-n)=\varphi(m)(-\varphi(-n))=\varphi(m)\varphi(n)$

If $m<0, n>0$: $m':=n$, $n':=m$ & do above $\quad$ QED

(7)(b) $\varphi(2)=1+1=0$. If m is even, $m=2k$ so $\varphi(m)=\varphi(2)\varphi(k)=0\cdot\varphi(k)=0$

If m is odd, $m=2k+1$ so $\varphi(m)=\varphi(2)\varphi(k)+\varphi(1)=0+1=1$ ✓

(c) Subspace test: let $\alpha\cdot w, \beta\cdot w \in \langle w\rangle$ Then $\alpha\cdot w+\beta\cdot w=(\alpha+\beta)\cdot w\in\langle w\rangle$

let $\gamma\in F$, Then $\gamma\cdot(\alpha\cdot w)=(\gamma\cdot\alpha)\cdot w\in\langle w\rangle$ QED

(8) (a) $W\leq V$: subspace test: let $w_1, w_2\in W$ so $w_1=(a_1, 2a_1, a_3, \dots a_n)$

& $w_2=(b_1, 2b_1, b_3, \dots b_n)$

Then $w_1+w_2=(a_1+b_1, 2(a_1+b_1), a_3+b_3, \dots, a_n+b_n)\in W$

If $\alpha\in F$, $\alpha w_1=(\alpha a_1, 2(\alpha a_1), \alpha a_3, \dots \alpha a_n)\in W$

(b) $W\not\leq V$: $(0,1,a_3,\dots a_n)+(0,1,a_3,\dots a_n)=(0,1+1,a_3+a_3,\dots a_n+a_n)$

& $1+1\neq 0+1$ so $\Rightarrow\notin W$

(c) $W\leq V$: subspace test: $(a_1, 2a_1, 3a_1, \dots na_1)+(b_1, 2b_1, \dots, nb_1)=(a_1+b_1, 2(a_1+b_1), \dots, n(a_1+b_1))\in W$

$\alpha(a_1, 2a_1, \dots, na_1)=(\alpha a_1, 2(\alpha a_1), \dots, n(\alpha a_1))\in W$

(d) Suppose $\exists \gamma_0\in F$ s.t. $2\gamma_0^2\neq 0$ Then $(\gamma_0,\gamma_0^2)+(\gamma_0,\gamma_0^2)=(\gamma_0+\gamma_0, \gamma_0^2+\gamma_0^2)$

& $(\gamma_0+\gamma_0)^2=\gamma_0^2+2\gamma_0^2+\gamma_0^2$

so $\gamma_0^2+\gamma_0^2\neq(\gamma_0+\gamma_0)^2$

so $\Rightarrow\notin W$.

so in $F=\mathbb{R}$, $W\not\leq V$

but in $F=\mathbb{F}_2$, $W\leq V$ (because really $a_1^2=a_1$)

(The full answer would be that $W\leq V$ iff $1+1=0$ in F. Can you see why?)

(e) $W\leq V$: subspace test: $f_1, f_2\in W$, Then $(f_1+f_2)(1)=f_1(1)+f_2(1)=0+0=0$ so $f_1+f_2\in W$

if $\alpha\in F$, $(\alpha\cdot f_1)(1)=\alpha\cdot f_1(1)=\alpha\cdot 0=0$ so $\alpha f_1\in W$ QED.

(f) $W\not\leq V$: let $f\in W$, Then $(f+f)(1)=f(1)+f(1)=1+1\neq 1$ (if $1+1=1$, then $1=0$)

(9)(a) ($\Rightarrow$) if $W=\bigcap_{\substack{U\leq V\\ S\subseteq U}} U$ & $W'\leq V$ is s.t. $S\subseteq W'$. Then W' is one of the U's in the intersection.

Thus $W=\bigcap_{\substack{U\leq V\\ S\subseteq U}} U\subseteq W'$.

($\Leftarrow$) Suppose $W\leq V$ is s.t. $S\subseteq W$ & $\forall W'\leq V$ with $S\subseteq W'$, we have $W\subseteq W'$. show $W=\bigcap_{\substack{U\leq V\\ S\subseteq U}} U$

(9)(a) (cont'd) by assumption $W \leq U$ for all U s.t. $U \leq V$ & $S \subseteq U$, Thus $W \leq \bigcap_{\substack{U \leq V \\ S \subseteq U}} U$

conversely, $W \leq V$ & $S \subseteq W$ so $\bigcap_{\substack{U \leq V \\ S \subseteq U}} U \leq W$.

So $W = \bigcap_{\substack{U \leq V \\ S \subseteq U}} U$ QED

(b) Part 1: induction on n: (i) base case: $n=2$: $v_1, v_2 \in V \Rightarrow v_1 + v_2 \in V$ is an axiom.

(ii) induction step: suppose $\forall v_1, \dots, v_{n-1} \in V$, have $v_1 + \dots + v_{n-1} \in V$

Then $V_1 + V_2 + \dots + V_n = \underbrace{(V_1 + \dots + V_{n-1})}_{\in V \text{ by induction hypothesis}} + V_n$

$= v + v_n \in V$ by axiom QED.

Part 2: $S \subseteq V$ & $W \leq V$ with $S \subseteq W$. let $\alpha_1, \dots, \alpha_n \in F$, $v_1, \dots, v_n \in S$

Since $v_i \in W$ & W is a vector space $\alpha_i v_i \in W$.

since $\alpha_i v_i \in W$ $\forall i$, by Part 1, $\alpha_1 v_1 + \dots + \alpha_n v_n \in W$. QED

(c) subspace test: let $v = \sum_{i=1}^{n} \alpha_i v_i$, $w = \sum_{j=1}^{m} \beta_j w_j$ be two lin combs, $\alpha_i, \beta_j \in F$, $v_i, w_j \in S$

Then $v + w = \alpha_1 v_1 + \dots + \alpha_n v_n + \beta_1 w_1 + \dots + \beta_n w_n$ is a lin comb of elements of S

If $\gamma \in F$, $\gamma v = \sum_{i=1}^{n} (\gamma \alpha_i) v_i$ is a lin comb of elements of S QED

(d) let U = set of all lin combs of elements of S. Then $U \leq V$ & $S \subseteq U$, so $\mathrm{Span}(S) \subseteq U$

By part (c), any $W \supseteq S$, contains U, so $U \subseteq \mathrm{Span}(S)$. so $\mathrm{Span}(S) = U$ QED

(10) ($\Leftarrow$) If $W_i \leq W_j$ Then $W_i \cup W_j = W_j \leq V$ ✓

($\Rightarrow$) If $W_1 \not\subseteq W_2$ & $W_2 \not\subseteq W_1$, Then $\exists w_1 \in W_1 \setminus W_2$ & $w_2 \in W_2 \setminus W_1$, so $w_1, w_2 \in W_1 \cup W_2$

claim: $w_1 + w_2 \notin W_1 \cup W_2$

proof: if $w_1 + w_2 \in W_1 \cup W_2$, Then $\exists v_i \in W_i$ s.t. $w_1 + w_2 = v_i$ for $i = 1$ or 2

wlog $i=1$. Then $w_2 = v_1 - w_1 \in W_1$. contradiction! QED