Math 411

Asst 3 Solutions

(1) Let S be a minimal generating set & suppose S is linearly dependent. Write $S=\{v_i : i\in I\}$
Then $\exists\, i_0 \in I$ s.t. $v_{i_0} = \sum_{\substack{i\neq i_0 \\ i\in I}} \alpha_i v_i$, $\alpha_i \in F$.

Let $S' = S \setminus \{v_{i_0}\}$. Since v_{i_0} is a lin comb. of the other $v_i \in S$, $\mathrm{Span}(S')=\mathrm{Span}(S)$.
So S' is a generating set. But $S' \subsetneq S$ contradicting the minimality of S. QED

(2) Suppose S is a maximal lin. indep set, i.e.

(i) S is lin. indep., and

(ii) $\forall S' \subseteq V$ s.t. S' is lin. indep. $S \not\subseteq S'$ (& $S\neq S'$)

Suppose $\mathrm{Span}(S) \neq V$. Then $\exists\, v \in V \setminus \mathrm{Span}(S)$. Let $S' = S \cup \{v\} \supsetneq S$. Write $S=\{v_i : i\in I\}$

Claim: S' is lin. indep.

proof: if $\beta v + \sum_{i\in I} \alpha_i v_i = 0$, & $\beta = 0$, then all $\alpha_i = 0$ since $\{v_i : i\in I\}$ is a basis

if $\beta \neq 0$, then $v = \sum_{i\in I}\left(\frac{-\alpha_i}{\beta}\right) v_i$ so $v \in \mathrm{Span}(S)$. Contradiction QED

Then S' is lin. indep. & $S' \supsetneq S$ contradicting the maximality of S. QED

(3) (a) Let $a,b,c \in \mathbb{Z}$.

(i) reflexivity: $a = a\cdot 1$ so $a|a$

(ii) antisymm: if $a=bd$ & $b=ae$ then $a=ade$, so $1=de$. Since $d,e \geq 1$ in $\mathbb{Z}$ get $d=e=1$

Thus $a = b\cdot 1$ so $a=b$

(iii) transitivity: if $a|b$ & $b|c$, then $b=ad$ & $c=be$ so $c=a\cdot(de)$ so $a|c$. QED

(b) $\forall a \in \mathbb{Z}_{\geq 1}$, $a = 1\cdot a$ so $1|a$ $\forall a \in \mathbb{Z}_{\geq 1}$. Also, if $a|1$, then $a=1$ (by antisymm)
so 1 is the unique minimal element.

(c) The primes!

(4) (a) If $0 \in S$, then $1\cdot 0 = 0$ is a non-triv. lin. comb. of elements of S that give 0.

(b) let $S' \subseteq S$. If $\exists\, v_1,\dots,v_n \in S'$ & $\alpha_i \in F$ s.t. $\sum_{i=1}^{n} \alpha_i v_i = 0$, this is still true if think of v_i as being in S. Thus $\alpha_i = 0$ $\forall i$

(4)(c) let $S' \supseteq S$. Then $\mathrm{Span}(S') \supseteq \mathrm{Span}(S)$, Also, $\mathrm{Span}(S') \subseteq V$ & $\mathrm{Span}(S) = V$
so $\mathrm{Span}(S') = V$

(d) let $S \subsetneq B$. let $v \in B \setminus S$.

Claim: $v \notin \mathrm{Span}(S)$

proof: if $v \in \mathrm{Span}(S)$, Then $S \cup \{v\}$ is lin dep. but $S \cup \{v\} \subseteq B$ & B is lin indep. so we get a contradiction from part (b).

let $S \supsetneq B$. let $v \in S \setminus B$. B is a basis so $v \in \mathrm{Span}(B)$ so $v \in \mathrm{Span}(S \setminus \{v\})$
so S is lin. dep. QED

(5) If $\alpha_1 \cdot 1 + \alpha_2 \cdot \sin(x) + \alpha_3 \cdot \cos(x) = 0$, Then, plugging in $x=0$, get that $\alpha_1 + \alpha_3 = 0$ so $\alpha_3 = -\alpha_1$

Plugging in $x = \frac{\pi}{2}$, get $\alpha_1 + \alpha_2 = 0$. So $\alpha_2 = -\alpha_1$.

Plugging in $x = \pi$, get $\alpha_1 - \alpha_3 = 0$. so $\alpha_1 = \alpha_3$. $\alpha_1 = \alpha_3 = -\alpha_1$ so $\alpha_1 = \alpha_3 = 0$. $\alpha_2 = -\alpha_1 = 0$

so $\alpha_1 = \alpha_2 = \alpha_3 = 0$ QED.

Claim: $\{1, \sin^2(x), \cos^2(x)\}$ is lin. dep.

proof: $1 \cdot 1 + (-1) \cdot \sin^2(x) + (-1) \cdot \cos^2(x) = 0$ (since $\sin^2(x) + \cos^2(x) = 1$ $\forall x \in \mathbb{R}$) QED.

(6) let S be a generating set of V s.t. $S \supseteq L$ & s.t. $\forall S' \subseteq V$, $S' \neq S$ with $S' \supseteq L$ & $\mathrm{Span}(S') = V$ have $S' \not\subseteq S$

Write $S = \{v_i : i \in I\} \cup L$ where $v_i \notin L$ $\forall i \in I$

Suppose S is lin. dep. Claim: $\exists i_0$ s.t. $v_{i_0} \in \mathrm{Span}(S \setminus \{v_{i_0}\})$

proof: S is lin dep so $\exists v \in S$ s.t. $v \in \mathrm{Span}(S \setminus \{v\})$

we're claiming we can pick v so that $v \notin L$. Suppose $v \in L$. Write $L = \{w_j : j \in J\}$

write $v = \sum_{i \in I} \alpha_i v_i + \sum_{\substack{j \in J \\ w_j \neq v}} \beta_j w_j$. Since L is lin indep, it must be that $\exists i_0 \in I$

s.t. $\alpha_{i_0} \neq 0$. Then $v_{i_0} = \frac{1}{\alpha_{i_0}} v + \sum_{\substack{i \in I \\ i \neq i_0}} \left(\frac{-\alpha_i}{\alpha_{i_0}}\right) v_i + \sum_{\substack{j \in J \\ w_j \neq v}} \left(\frac{-\beta_j}{\alpha_{i_0}}\right) w_j$.

i.e. $v_{i_0} \in \mathrm{Span}\{S \setminus \{v_{i_0}\}\}$ & $v_{i_0} \notin L$, as desired. QED

Let $S' = S \setminus \{v_{i_0}\}$. Then $S' \supseteq L$ & S' spans V & $S' \subsetneq S$ contradicting the minimality of S. QED.

(7) (a) If $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$, then $\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ so $\alpha_i = 0 \ \forall i$

If $\alpha_1 w_1 + \alpha_2 w_2 + \alpha_3 w_3 = 0$, then $\begin{pmatrix} \alpha_1 + \alpha_2 + \alpha_3 \\ \alpha_1 + \alpha_2 \\ \alpha_1 + \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

so $\alpha_1 + \alpha_2 = 0 \Rightarrow \alpha_1 + \alpha_2 + \alpha_3 = \alpha_3 = 0$

so $\alpha_3 = 0$

so $\alpha_1 + \alpha_3 = 0 \Rightarrow \alpha_1 = 0$ so $\alpha_2 = 0$ QED

(b) $w_1 = v_1 + v_2 + v_3$ so $v_1 = +w_1 - v_2 - v_3$ so can substitute w_1 for v_1

get $\{w_1, v_2, v_3\}$ is a basis of F^3

$w_2 = w_1 - v_3$ so $v_3 = w_1 - w_2$ so can substitute w_2 for v_3

get $\{w_1, w_2, v_2\}$ is a basis of F^3 (<u>Note</u>: w_1, w_2, w_3 is <u>not</u> a basis)

$w_3 = w_1 - v_2$ so $v_2 = w_1 - w_3$ so can substitute w_3 for v_2

get $\{w_1, w_2, w_3\}$ is a basis for F^3