Math 411

Asst. 4 Solutions

(1)(a) V has a basis $\mathcal{B}=\{v_1,v_2\}$, Then $V=\{\alpha_1 v_1+\alpha_2 v_2 : \alpha_1,\alpha_2\in\mathbb{F}_2\}$
2 elements in $\mathbb{F}_2$, each has two possibilites, so $\#V=2^2=\boxed{4}$

(b) 0-dim'l: 1 subspace $(\{0\})$
1-dim'l: 1 for each non-zero $v\in V$, so $4-1=3$ subspaces
2-dim'l: 1 subspace (V itself)
so $\boxed{\text{5 subspaces}}$

(c) To build a basis, pick any non-zero v_1 (3 possibilites)
~~For each~~ Then, pick a $v_2\notin \text{Span}(v_1)$.
1-dim'l & has 0 & v_1 in it only so 2 possibilities
so $3\cdot 2=\boxed{\text{6 bases}}$

(d) First pick any non-zero v_1 ($2^3-1=7$ possibilities, since $\#V=2^3=8$)
Then, pick any v_2 not in $\text{span}(v_1)=\{0,v_1\}$ (so $2^3-2=6$ possibilities)
Then, pick any $v_3\notin\text{Span}(v_1,v_2)=\{0,v_1,v_2,v_1+v_2\}$ (so $2^3-4=4$ possibilities)
so $7\cdot 6\cdot 4=\boxed{\text{168 bases}}$

(e) let V be 2-dim'l $/\mathbb{F}_3$. $V=\{\alpha_1 v_1+\alpha_2 v_2 : \alpha_i\in\mathbb{F}^3\}$ so $\#V=3^2=9$

To find a basis of V, first pick any non-zero vector v_1 ($9-1=8$ possibilities)
Then, pick any $v_2\notin\text{span}(v_1)=\{0,v_1,2v_1\}$ (so $9-3=6$ possibilities)
so $8\cdot 6=\boxed{\text{48 bases}}$

(2)(a) let $E_{ij}=(e_{k\ell})$ where $e_{k\ell}=\begin{cases}1 & i=k,\ j=\ell\\ 0 & \text{otherwise}\end{cases}$ (For $m=n=2$: $E_{11}=\begin{pmatrix}1&0\\0&0\end{pmatrix}$, $E_{12}=\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $E_{21}=\begin{pmatrix}0&0\\1&0\end{pmatrix}$, $E_{22}=\begin{pmatrix}0&0\\0&1\end{pmatrix}$)

Claim: $\{E_{ij},\ i=1,\dots,m,\ j=1,\dots,n\}$ is a basis of $M_{m,n}(F)$
pf: It spans: let $A=(a_{ij})\in M_{m,n}(F)$. Then, clearly, $A=\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}E_{ij}$
It's lin. indp: Suppose $\alpha_{ij}\in F$ are s.t. $\sum_{i=1}^{m}\sum_{j=1}^{n}\alpha_{ij}E_{ij}=0$
Then the matrix (α_{ij}) is zero. so $\alpha_{ij}=0$ $\forall i,j$. QED

(2) (b) From part (a), $\dim M_{m,n}(F) = mn$.

(Assume $m=n$, throughout)

(3) (a) Let $A_1, A_2 \in S$ so $A_1^T = A_1$ & $A_2^T = A_2$. Let $A_1 = (a_{ij})$, $A_2 = (b_{ij})$

Note that $(A_1+A_2)^T = A_1^T + A_2^T$ (Indeed, $(A_1+A_2) = (a_{ij}+b_{ij})$

so $(A_1+A_2)^T = (a_{ji}+b_{ji}) = (a_{ji}) + (b_{ji})$)

so $(A_1+A_2)^T = A_1^T + A_2^T = A_1 + A_2$ ✓

Similarly, $(\lambda A_1)^T = \lambda A_1^T = \lambda A_1$ ✓

Let $A_1, A_2 \in \mathcal{A}$, so $A_1^T = -A_1$ & $A_2^T = -A_2$.

Again, $(A_1+A_2)^T = A_1^T + A_2^T = -A_1 - A_2 = -(A_1+A_2)$ ✓

& $(\lambda A_1)^T = \lambda A_1^T = -\lambda A_1 = -(\lambda A_1)$ ✓

(b) <u>Claim 1</u>: $S + \mathcal{A} = M_{m,n}(F)$, if $1+1 \neq 0$ in F. (so that $2^{-1} \in F$)

proof: let $A \in M_{m,n}(F)$. Let $A_+ := \frac{A+A^T}{2}$, $A_- := \frac{A-A^T}{2}$

Then $A_+^T = \left(\frac{A+A^T}{2}\right)^T = \frac{A^T+A}{2} = A_+ \in S$

& $A_-^T = \left(\frac{A-A^T}{2}\right)^T = \frac{A^T-A}{2} = -A_-$ so $A_- \in \mathcal{A}$

& $A = \frac{A+A^T}{2} + \frac{A-A^T}{2} = A_+ + A_-$ ✓ QED

<u>Claim 2</u>: $S \cap \mathcal{A} = 0$, if $1+1 \neq 0$ in F

proof: if $A \in S$, then $a_{ij} = a_{ji}$

if $A \in \mathcal{A}$, then $a_{ij} = -a_{ji}$ so $a_{ij} = -a_{ij}$ $\forall i,j$.

so $(1+1)a_{ij} = 0$. but $1+1 \neq 0$, so $a_{ij} = 0$ $\forall i,j$ QED

(And, in fact the question is incorrect whenever $1+1=0$ in F. Indeed, if $1+1=0$, then $\alpha = -\alpha$ $\forall \alpha \in F$, so $\mathcal{A} = S$ so $\mathcal{A} \cap S \neq 0$ & $\mathcal{A} + S \neq M_{m,n}(F)$ (unless $m=n=1$): the matrix (a_{ij}) with $a_{ij} = 1$ $\forall i,j$ is in $\mathcal{A} \cap S$. Also, the matrix with $a_{ij} = 1$ $\forall i,j$ except $i=2, j=1$, is not in $\mathcal{A} = \mathcal{A}+S$.)

(c) Yes $\mathcal{U}$ is a subspace: let $u_1, u_2 \in \mathcal{U}$. $u_1 = (a_{ij})$, $u_2 = (b_{ij})$

let $u_1 + u_2 = (c_{ij})$. $c_{ij} = a_{ij} + b_{ij}$ & if $j \le i$ $c_{ij} = 0 + 0 = 0$. ✓

also, $\lambda u_1 = (\lambda a_{ij})$ & $\lambda a_{ij} = \lambda \cdot 0 = 0$ if $j \le i$

Claim 1: $M_{n,n}(F) = S \oplus \mathcal{U}$

proof: $M_{n,n}(F) = S + \mathcal{U}$: let $A \in M_{n,n}(F)$. Let $A = (a_{ij})$ & define $A' = (a'_{ij})$ where $a'_{ij} = \begin{cases} a_{ij} & \text{if } j \le i \\ a_{ji} & \text{if } j > i \end{cases}$ (if $i = j$, then $a_{ij} = a_{ji}$)

Then $A' \in S$.

Define $U = (u_{ij})$ by $A - A'$.

Then $U \in \mathcal{U}$ because $a_{ij} - a'_{ij} = 0$ for $j \le i$.

so $A = A' + U$ ✓

$S \cap \mathcal{U} = 0$: if $A = (a_{ij}) \in S$, then $a_{ij} = a_{ji}$ $\forall i, j$.

if $A \in \mathcal{U}$, then $a_{ij} = 0$ for $j \le i$

if $i > j$, then $a_{ij} = a_{ji} = 0$ (since $i \le j$)

so $A = 0$ QED

Claim 2: $M_{n,n}(F) = A \oplus \mathcal{U}$ iff $1+1=0$ in F

proof: If $1+1=0$ in F, then $A = S$, so Claim 1 shows that $M_{n,n}(F) = A \oplus \mathcal{U}$

If $1+1 \ne 0$ in F, then $1 \ne -1$, so $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \notin A$. Also $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \notin \mathcal{U}$.

so $M_{n,n}(F) \ne A + \mathcal{U}$. QED

(4) Let $W_1, W_2 \le \mathbb{R}^3$ with $\dim(W_i) = 2$, then $\dim(W_1 + W_2) = 2$ or 3 ≤ 3

so $\dim(W_1 \cap W_2) = -\dim(W_1 + W_2) + \dim(W_1) + \dim W_2 \ge -3 + 2 + 2 = 1$ QED

(5) Neither P nor N are subspaces: Let $f(x) = 1$ for all x. Then $-f(x) \notin P$.

Also $-f(x) \in N$, but $-(-f(x)) = f(x) \notin N$.

(This question should say that $W_i \neq 0$ for all i)

(6)(a) Let $n = \dim(V)$. Suppose $\# I > n$. We saw in class that $\forall v \in V$, $\exists$ unique $w_i \in W_i$ s.t. $v = \sum_{i \in I} w_i$.

Let $W_1, \dots, W_{n+1}$ be $n+1$ of the subspaces in $\{W_i : i \in I\}$ & let $w_i \in W_i$ for $i = 1, \dots, n+1$, be non-zero vectors

since $n+1 > \dim(V)$, $\exists \alpha_i \in F$, $i = 1, \dots, n$ s.t. $w_{n+1} = \sum_{i=1}^{n} \alpha_i w_i$ (up to relabelling $w_1, \dots, w_{n+1}$)

but then $w_{n+1} = w_{n+1}$

& $w_{n+1} = \sum_{i=1}^{n} \alpha_i w_i$ are two distinct ways to write w_{n+1}. contradiction.

Hence $\# I \leq n < \infty$. QED

(b) We saw that if $W_1, W_2 \leq V$ & $\dim(V) < \infty$, & $V = W_1 \oplus W_2$, then $\dim V = \dim W_1 + \dim W_2$

Let $W_1, \dots, W_m \leq V$ with $\bigoplus_{i=1}^{m} W_i = V$.

Claim: $\dim V = \sum_{i=1}^{m} \dim W_i$

proof: by induction on m:

(i) base case: $m=2$: done in class

(ii) induction step: ind. hypothesis: it's true for $m-1$ subspaces.

let $W = \bigoplus_{i=1}^{m-1} W_i$. Then by ind. hypo.: $\dim W = \sum_{i=1}^{m-1} \dim W_i$

Claim: $W \oplus W_m = V$

proof: Since $V = \bigoplus_{i=1}^{m} W_i$, $V = W + W_m$

so show $W_m \cap W = 0$. write $w = w_1 + \dots + w_{m-1} \in W$, $w_i \in W_i$

Let $w \in W_m \cap W$, $w \neq 0$. Then $w = \overbrace{0 + 0 + \dots + 0}^{m-1} + w$

& $w = w_1 + w_2 + \dots + w_{m-1} + 0$

are two distinct ways to write w in $\bigoplus_{i=1}^{m} W_i$, contradicting that $\bigoplus_{i=1}^{m} W_i$ is a direct sum. ✓

Then $\dim = \dim W + \dim W_m = \sum_{i=1}^{m} W_i$ QED.

(7) Claim 1: $V = \sum_{b \in B} W_b$

prof: Let $v \in V$. $\exists \alpha_b \in F$ s.t. $v = \sum_{b \in B} \alpha_b b$ since B is a basis.

~~$\alpha_b b$~~ $\alpha_b b \in W_b$ so This shows $V = \sum_{b \in B} W_b$. QED

Claim 2: $\forall b \neq b'$ in B, $W_b \cap W_{b'} = 0$.

prof: $W_b = \{\alpha b : \alpha \in F\}$. If $W_b \cap W_{b'} \not\ni 0$, Then there's a nonzero vector $\alpha_1 b = \alpha_2 b' \in W_b \cap W_{b'}$. Then $\alpha_1 b - \alpha_2 b' = 0$ with $\alpha_1 \neq 0 \neq \alpha_2$ so b & b' are lin. dependent. Contradiction. QED

(8) (a) $\dim V = \dim W + \dim W'$ so $\boxed{\dim W' = \dim V - \dim W} \boxed{= n-d}$

(b) let W' = The z-axis. $W + W' = \mathbb{R}^3$: $(x,y,z) = (x,y,0) + (0,0,z)$

$W \cap W' = 0$: if $x=y=0$ & $z=0$, Then $(x,y,z) = (0,0,0)$

let $W'' = \mathrm{Span}((1,1,1))$. $W + W'' = \mathbb{R}^3$ $(x,y,z) = (x-z, y-z, 0) + (z,z,z)$

$W \cap W'' = 0$: if $z=0$ (ie. if $(x,y,z) \in$ xy-plane)

& $(x,y,z) \in W''$ (ie. $(x,y,z) = \alpha(1,1,1)$)

Then $\alpha(1,1,1) = (\alpha,\alpha,\alpha) = (x,y,0)$

so $\alpha = 0$, so $\alpha(1,1,1) = 0$.

(c)

(9) No, because $W' \not\leq V$: let $f_1(x) = \begin{cases} 1 & \text{if } x \neq \pi \\ 0 & \text{if } x = \pi \end{cases}$, let $f_2(x) = \begin{cases} 1 & \text{if } x \neq e \\ 0 & \text{if } x = e \end{cases}$

Then $f_1, f_2 \in W'$. But $f_1 - f_2 \notin W'$, since $(f_1 - f_2)(x) = 0$ for all $x \neq \pi, e$.

$W \leq V$: if $f = \sum_{i=0}^{d} a_i x^i$ & $g = \sum_{i=0}^{d} b_i x^i$, Then $f+g = \sum_{i=0}^{d} (a_i + b_i) x^i$ & $\lambda f = \sum_{i=0}^{d} \lambda a_i x^i \in W$

let $W' := \mathrm{Span}\{x^{d+1}, x^{d+2}, \ldots\} = \{\sum_{i=d+1}^{\infty} a_i x^i\}$

$V = W + W'$: $\sum_{i=0}^{\infty} a_i x^i = \sum_{i=0}^{d} a_i x^i + \sum_{i=d+1}^{\infty} a_i x^i \in W + W'$

$W \cap W' = 0$: $f \in W \cap W'$, Then $f = \sum_{i=0}^{d} a_i x^i = \sum_{i=d+1}^{\infty} a_i x^i$ so $a_i = 0$ $\forall i$. QED