Math 411

Asst. 5 solutions

(1)(a) let ~~[illegible]~~ S be the set of sol'ns to a sys. of lin equations over $\mathbb{R}$, & let S_0 be the set of sol'ns of associated homog." - - - - - - - .

If $v \in S$, then $S = v + S_0 = \{v + w : w \in S_0\}$.

Now S_0 is a vector space & so has a dim. If $\dim S_0 = 0$, then $\#S = 1$, since $S = \{v\}$

If $\dim S_0 > 0$, then there is a non-zero vector $w_0 \in S_0$, so $\alpha w_0 \in S_0$ $\forall \alpha \in \mathbb{R}$ & $\alpha w_0 \neq \beta w_0$ for $\alpha \neq \beta$, so $\#S = \infty$

So if $\exists v \in S$, (ie. if $\#S \neq 0$), then $\#S = 1$ or ∞.

System with no sol'ns: $\begin{cases} x_1 = 0 \\ 0 = 1 \end{cases}$ (or simply $\{0 = 1$)

System with one sol'n: $\begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases}$ (or simply $\{x_1 = 0$)

System with ∞ly many sol'ns: $\{x_1 + x_2 = 0$ (or simply $0 = 0$)

(b) As above, $S = v + S_0$, if $\exists v \in S$. Since $\dim S_0 = \#$ of free variables $\leq \#$ variables $= 2$

$\dim S_0 = 0, 1,$ or 2, so either $\#S = 0$, or $\#S = 2^{\dim S_0} = 1, 2,$ or 4.

(since $\#F^n = 2^n = 2^{\dim F^n}$)

0 solutions: $\{0 = 1$

1 solution: $\begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases}$

2 solutions: $\{x_1 + x_2 = 0$

4 solutions: $\{0 = 0$

(2) If there are 4 variables & two eqns then there are at most two pivots so at least 2 free variables.

so dim of sol'n space is ≥ 2. So **no!**

(Alternatively, $rk + \text{nullity} = 4$ & $rk \leq 2$ so $\text{nullity} \geq 2$)

(3) (a) $Sol(S) = 2$ dim'l & $Sol(S') = 5$ dim'l. $Sol(\text{combined sys}) = Sol(S) \cap Sol(S')$

Denote it by V. Then $\dim V = \dim(Sol(S)) + \dim(Sol(S')) - \dim(Sol(S) + Sol(S'))$

depending on the overlap of $Sol(S)$ & $Sol(S')$, dim of their sum is between 5 & $5 + 2 = 7$

So $\dim V = 7 - (5 \text{ or } 6 \text{ or } 7) = 0, 1,$ or 2.

(3)(b) If there are only six variables, then $\dim(\mathrm{Sol}(S) + \mathrm{Sol}(S')) \le 6$, so $\dim V \ge 7-6=1$, so $\exists$ non-zero v that is a sol'n to both systems

(4)
$$\begin{pmatrix} 0&0&0&5&0 \\ 0&2&4&0&6 \\ 0&-2&-4&3&-8 \end{pmatrix} \xrightarrow[\substack{R_1 \to \frac{R_1}{5} \\ R_2 \to \frac{R_2}{2} \\ R_1 \leftrightarrow R_2}]{R_3 \to R_3 + R_2} \begin{pmatrix} 0&1&2&0&3 \\ 0&0&0&1&0 \\ 0&0&0&3&-2 \end{pmatrix} \xrightarrow[R_3 \to \frac{R_3}{-2}]{R_3 \to R_3 - 3R_2} \begin{pmatrix} 0&1&2&0&3 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{pmatrix} \xrightarrow{R_1 \to R_1 - 3R_3} \begin{pmatrix} 0&1&2&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{pmatrix}$$

$\uparrow$ x_1 $\quad \uparrow$ x_3

so basis: $\boxed{\begin{pmatrix} 1\\0\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\-2\\1\\0\\0 \end{pmatrix}}$

(5)
$$\left(\begin{array}{ccc|c} 1&2&-1&4 \\ 1&4&2&6 \\ 1&4&?&-1 \end{array}\right) \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \left(\begin{array}{ccc|c} 1&2&-1&4 \\ 0&2&3&2 \\ 0&2&?+1&-5 \end{array}\right) \xrightarrow{R_3 \to R_3 - R_2} \left(\begin{array}{ccc|c} 1&2&-1&4 \\ 0&2&3&2 \\ 0&0&?-2&-7 \end{array}\right)$$

so $?-2=0$ so $\boxed{?=2}$

(6)
$$\mathrm{Span}\left(\begin{pmatrix} 1\\1\\0\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\5\\-1\\1\\0 \end{pmatrix}, \begin{pmatrix} -4\\0\\0\\6\\0\\2 \end{pmatrix}\right) = \mathrm{Span}\left(\begin{pmatrix} 1\\1\\0\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\5\\-1\\1\\0 \end{pmatrix}, \begin{pmatrix} -2\\0\\0\\3\\0\\1 \end{pmatrix}\right)$$

The free variables are x_2, x_5, & x_6, so three pivots, so matrix needs at least 3 rows.

The augmented matrix is $(A\ 0)$ where $\boxed{A = \begin{pmatrix} 1&-1&0&0&0&2 \\ 0&0&1&0&-5&0 \\ 0&0&0&1&1&-3 \end{pmatrix}}$

(7)
$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ 0 = 1 \end{cases}$$

(8)(a) Let $W = \{b \in F^m : (A\ b) \text{ has a sol'n}\}$. Let $b, b' \in W$. Write $A = (a_{ij})$, $b = (b_i)$, $b' = (b'_i)$

Then $\exists$ $x = (x_1, \ldots, x_n)$ & $x' = (x'_1, \ldots, x'_n)$ s.t. $\sum_{j=1}^{n} a_{ij} x_j = b_i$ & $\sum_{j=1}^{n} a_{ij} x'_j = b'_i$ for all $i = 1, \ldots, m$. Then $\sum_{j=1}^{n} a_{ij}(x_j + x'_j) = b_i + b'_i$ so $x_j + x'_j$ is a sol'n to $(A\ b+b')$

so $b + b' \in W$.

Similarly, $\sum_{j=1}^{n} a_{ij} \lambda x_j = \lambda b_i$, so $\lambda b \in W$. QED

(see next page for a more conceptual solution)

(a) Alternate sol'n:

The sys $(A\ b)$ has a sol'n iff $\exists \alpha_1, \ldots, \alpha_n \in F$ s.t.

$$\alpha_1 c_1 + \cdots + \alpha_n c_n = b$$

where c_j are the columns of A.

So $W = \mathrm{Col}(A)$! QED

(b) The space of such b is the $\mathrm{Col}(A)$. So row reduce & take pivot cols of A.

$$\begin{pmatrix} 1 & 2 & 2 \\ 3 & 6 & 1 \\ 1 & 2 & 1 \\ -1 & -2 & 3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & 2 \\ 0 & 0 & -2 \\ 0 & 0 & -1 \\ 0 & 0 & 5 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So $\boxed{\begin{pmatrix} 1 \\ 3 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix} \text{ is a basis}}$