Math 411

Asst. 6 Solutions

(1) (a) $\begin{pmatrix} 1 & 2 & 0 & 1 \\ 1 & 2 & 2 & 7 \\ 3 & 6 & 2 & 9 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 2 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

so $\boxed{\text{basis of } Col(A) = \left\{ \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} \right\}}$

& $\boxed{\text{basis of } Row(A) = \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \end{pmatrix} \right\}}$

& $\boxed{\text{basis of } Nul(A) = \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -3 \\ 1 \end{pmatrix} \right\}}$

$x_1 = -2x_2 - x_4$
$x_2 = x_2$
$x_3 = -3x_4$
$x_4 = x_4$

(b) Same bases because row-reducing gives the same thing.

(c) " — — — — — — — — — — — — .

(d) Here, row-reduction is a bit different because $2 = 6 = 0$, so

$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 1 & 2 & 2 & 7 \\ 3 & 6 & 2 & 9 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

so basis of $Col(A) = \left\{ \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \right\}$

basis of $Row(A) = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

basis of $Nul(A) = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

(2) (a) Let $A = \begin{pmatrix} 1 & 1 & 3 & 4 \\ -3 & 3 & -3 & -6 \\ -3 & 9 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 3 & 4 \\ 0 & 6 & 6 & 6 \\ 0 & 12 & 12 & 12 \\ 0 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 3 & 4 \\ 0 & 6 & 6 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Then $W = Col(A)$ so $\boxed{\{v_1, v_2, v_4\} \text{ is a basis of } W.}$

(2)(b) let $B = \begin{pmatrix} 1 & -3 & -3 & 0 \\ 1 & 3 & 9 & 0 \\ 3 & -3 & 3 & 0 \\ 4 & -6 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -3 & -3 & 0 \\ 0 & 6 & 12 & 0 \\ 0 & 6 & 12 & 0 \\ 0 & 6 & 12 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -3 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

so $Row(B) = W$. So $\boxed{\left\{ \begin{pmatrix} 1 \\ 0 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}}$ is a reduced echelon basis of W.

(3) (a) if A' is in REF, Then bringing to RREF involves (1) multiplying rows by non-zero scalars to make the leading entries equal to 1. This doesn't affect which entries are 0 versus not 0, so doesn't change the locations of the leading entries. The other operation is (2) adding a multiple of R_j to R_i for $j > i$ to clear the non-zero entries above a leading entry. Since $j > i$, The leading entry of R_i is unaffected; indeed, the kth entry of R_j is 0 for all $k \le i$, so the first i entries of R_i are unaffected. QED

(b) $Row(A') = Row(A) = Row(RREF(A))$. The proof we gave in class showing that the non-zero rows of $RREF(A)$ are a basis of $Row(A)$ works as is for A'.

(4) (a) Let $v_1, \dots, v_d$ be the given basis of $Nul(A)$, let w_{d+1-t} = "reverse" of v_t, $t = 1, 2, \dots, d$ (ie. if $v_t = \begin{pmatrix} a_{t1} \\ a_{t2} \\ \vdots \\ a_{tn} \end{pmatrix}$ then $w_{d+1-t} = \begin{pmatrix} a_{tn} \\ a_{t,n-1} \\ \vdots \\ a_{t1} \end{pmatrix}$). Note $w_i \ne 0$ $\forall i$ since any set containing 0 is lin dep & $v_1, \dots, v_d$ is a basis.

Claim 1: The first non-zero entry ~"top most" of w_{d+1-t} is 1.

proof: for $k > j_t$, $a_{tk} = 0$ & $a_{tj_t} = 1$. QED

Claim 2: The "top most" non-zero entry of w_{d+1-t} is above that of "$w_{d+1-t+1}$" $w_{d+1-(t-1)}$

proof: $j_{t-1} < j_t$ so $a_{t-1,k} = 0$ for $k \ge j_t$ QED

~~Claim 3: ... Note that the leading entry of w_{d+1-t} is the entry~~

(4)(a) (cont'd)

Claim 3: Placing $w_1, w_2, \ldots w_d$ as the rows of a matrix A, the leading entry of row r is the only non-zero entry in its column.

proof: let $t = d+1-r$, then the leading entry of row r is in column $d+1-j_t$.

For $s \neq r$, the $d+1-j_t$ entry of w_s is the j_t entry of v_{d+1-s}.

Now $d+1-s \neq d+1-r = t$ so the j_t entry of $v_{d+1-s} = 0$ QED.

These 3 claims show that A is in RREF.

(b) let $C = \begin{pmatrix} 0 & -3 & -3 & 1 \\ 0 & 9 & 3 & 1 \\ 0 & 3 & -3 & 3 \\ 1 & 0 & -6 & 4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & -6 & 4 \\ 0 & 3 & -3 & 3 \\ 0 & 0 & 12 & 10 \\ 0 & 0 & 12 & 10 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & -6 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 5/6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\downarrow$

$\begin{pmatrix} 1 & 0 & 0 & 9 \\ 0 & 1 & 0 & 11/6 \\ 0 & 0 & 1 & 5/6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

so $\mathrm{Span}\{v_1, v_2, v_3, v_4\} = \mathrm{Span}\left\{ \begin{pmatrix} 9 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 11/6 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 5/6 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$

so let $A = \begin{pmatrix} 1 & -5/6 & -11/6 & -9 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Then $\mathrm{Nul}(A) = \mathrm{Span}\{v_1, \ldots, v_4\}$

(5) Consider $1 \in \mathbb{R}$. If $T(1) = 0$, then $\forall b \in \mathbb{R}$, $T(b) = T(b \cdot 1) = b \cdot T(1) = 0$

so $T = T_\alpha$ for $\alpha = 0$.

Say $T(1) \neq 0$, let $\alpha = T(1)$, then $\forall b \in \mathbb{R}$ $T(b) = T(b \cdot 1) = b \cdot T(1) = b \cdot \alpha = \alpha b$

so $T = T_\alpha$ for $\alpha = T(1)$ (in all cases). QED

(6) let $b, b' \in F^n$, then $(A(b+b'))_i = \sum_{k=1}^{n} A_{ik}(b_k + b'_k) = \sum_{k=1}^{n} A_{ik} b_k + \sum_{k=1}^{n} A_{ik} b'_k$

$= (Ab)_i + (Ab')_i$ ✓

If $\lambda \in F$, then $(A(\lambda b))_i = \sum_{k=1}^{n} A_{ik}(\lambda b_k) = \lambda \sum_{k=1}^{n} A_{ik} b_k = \lambda (Ab)_i$ ✓

QED

(7) (i) $T(0) = T(0+0) = T(0) + T(0)$ add $-T(0)$ to both sides.

so $0 = T(0)$ ✓

(ii) $T(v) + T(-v) = T(v+(-v)) = T(0) = 0$

so $T(-v) = -T(v)$ ✓

(iii) For $v_1, v_2 \in V'$ & $\lambda \in F$, $T|_{V'}(v_1+v_2) = T(v_1+v_2) = T(v_1) + T(v_2) = T|_{V'}(v_1) + T|_{V'}(v_2)$

& $T|_{V'}(\lambda v_1) = T(\lambda v_1) = \lambda T(v_1) = \lambda T|_{V'}(v_1)$ ✓

(iv) let $T(v_1), T(v_2) \in T(V')$ (so $v_1, v_2 \in V'$)

Then $T(v_1) + T(v_2) = T(v_1 + v_2) \in T(V')$ since $v_1 + v_2 \in V'$ ✓

let $\lambda \in F$. $\lambda v_1 \in V'$ so $\lambda T(v_1) = T(\lambda v_1) \in T(V')$ ✓

(v) let $v_1, v_2 \in \ker(T)$, Then $T(v_1+v_2) = T(v_1) + T(v_2) = 0 + 0 = 0$ ✓

& for $\lambda \in F$, $T(\lambda v_1) = \lambda T(v_1) = \lambda \cdot 0 = 0$ ✓

~~Suppose not~~:

(vi) let $\alpha_1, \ldots \in F$ s.t. ~~$\sum_{i=1}^{n} \alpha_i T(v_i) = 0$~~.

$\sum_{i \in I} \alpha_i v_i = 0$.

Then $T\left(\sum_{i \in I} \alpha_i v_i\right) = T(0) = 0$

$= \sum_{i \in I} \alpha_i T(v_i)$. Now the $T(v_i)$ are lin. indep so $\alpha_i = 0 \ \forall i$. QED

(vii) Let $w \in \operatorname{im}(T)$. Then $\exists\, v \in V$ s.t. $T(v) = w$.

$\exists\, \alpha_1, \ldots, \alpha_n \in F$ & $v_1, \ldots, v_n \in \{v_i : i \in I\}$ s.t. $\sum_{i=1}^{n} \alpha_i v_i = v$.

Then $w = T(v) = T\left(\sum_{i=1}^{n} \alpha_i v_i\right) = \sum_{i=1}^{n} \alpha_i T(v_i)$

so $w \in \operatorname{Span}\{T(v_i) : i \in I\}$ i.e. $\operatorname{im}(T) = \operatorname{Span}\{T(v_i) : i \in I\}$ QED