Math 411

Asst 7 Solutions

(1) (a) For $v_1, v_2 \in V$, $(T \circ T')(v_1+v_2) = T(T'(v_1+v_2)) = T(T'(v_1)+T'(v_2))$

$$= T(T'(v_1)) + T(T'(v_2))$$

$$= (T\circ T')(v_1) + (T\circ T')(v_2)$$

For $\lambda \in F, v \in V$, $(T\circ T')(\lambda v) = T(T'(\lambda v)) = T(\lambda T'(v)) = \lambda T(T'(v)) = (\lambda(T\circ T'))(v)$

QED

(b) $(T_1+T_2)(v_1+v_2) = T_1(v_1+v_2) + T_2(v_1+v_2) = T_1(v_1) + T_1(v_2) + T_2(v_1) + T_2(v_2)$

$$= (T_1+T_2)(v_1) + (T_1+T_2)(v_2)$$

$(T_1+T_2)(\lambda v) = T_1(\lambda v) + T_2(\lambda v) = \lambda T_1(v) + \lambda T_2(v)$

$$= \lambda(T_1(v) + T_2(v)) = \lambda((T_1+T_2)(v))$$ QED

(c) $(\lambda T)(v_1+v_2) = \lambda \cdot T(v_1+v_2) = \lambda \cdot (T(v_1)+T(v_2)) = \lambda \cdot T(v_1) + \lambda \cdot T(v_2)$

$$= (\lambda T)(v_1) + (\lambda T)(v_2)$$

$(\lambda T)(\alpha v) = \lambda T(\alpha v) = \lambda \cdot \alpha \cdot T(v) = \alpha \cdot \lambda \cdot T(v) = \alpha \cdot (\lambda T)(v)$ QED

(2) (i) Assoc of +: $T_i \in \mathrm{Hom}(V,W)$ $i = 1,2,3$, then $(T_1 + (T_2+T_3))(v) = T_1(v) + (T_2+T_3)(v)$

$$= T_1(v) + (T_2(v) + T_3(v))$$

$$= (T_1(v) + T_2(v)) + T_3(v)$$

$$= (T_1+T_2)(v) + T_3(v)$$

$$= ((T_1+T_2)+T_3)(v)$$ ✓

(ii) comm. of +: $(T_1+T_2)(v) = T_1(v) + T_2(v) = T_2(v) + T_1(v) = (T_2+T_1)(v)$ ✓

(iii) Add. id: $(0+T)(v) = 0(v) + T(v) = 0 + T(v) = T(v)$ ✓

(iv) Add inv: Given $T \in \mathrm{Hom}(V,W)$, let $-T : V \to W$ given by $(-T)(v) = -(T(v))$

Claim: Then $-T$ is linear:

proof: $(-T)(v_1+v_2) = -(T(v_1+v_2)) = -T(v_1) - T(v_2) = (-T)(v_1) + (-T)(v_2)$

$(-T)(\lambda v) = -(T(\lambda v)) = -\lambda T(v) = \lambda(-T)(v)$ QED

Claim: $T + (-T) = 0$

proof: $(T+(-T))(v) = T(v) + (-T(v)) = 0$ QED ✓

(v) "assoc" of $\cdot$: $(\alpha \cdot (\beta \cdot T))(v) = \alpha \cdot (\beta \cdot T)(v) = \alpha \cdot \beta \cdot T(v) = ((\alpha \cdot \beta) \cdot T)(v)$ ✓

(2)(vi) ~~Distr.~~ Distr. part 1: $((\alpha+\beta)\cdot T)(v) = (\alpha+\beta)\cdot T(v) = \alpha\cdot T(v) + \beta\cdot T(v)$

$= (\alpha\cdot T)(v) + (\beta\cdot T)(v)$ ✓

(vii) Dist. part 2: $(\alpha\cdot(T_1+T_2))(v) = \alpha\cdot(T_1+T_2)(v) = \alpha\cdot(T_1(v)+T_2(v))$

$= \alpha\cdot T_1(v) + \alpha\cdot T_2(v)$

$= (\alpha\cdot T_1)(v) + (\alpha\cdot T_2)(v)$ ✓

(vii) mult. id: $(1\cdot T)(v) = 1\cdot T(v) = T(v)$ ✓

* Note: The question is incorrect as stated: $\bigoplus_{y\in Y} W$ should be $\prod_{y\in Y} W$.

(3)(a)* $T_Y(f_1+f_2) = ((f_1+f_2)(y))_{y\in Y} = (f_1(y)+f_2(y))_{y\in Y}$

$= (f_1(y))_{y\in Y} + (f_2(y))_{y\in Y}$

$= T_Y(f_1) + T_Y(f_2)$ ✓

$T_Y(\lambda f) = ((\lambda f)(y))_{y\in Y} = (\lambda f(y))_{y\in Y} = \lambda(f(y))_{y\in Y} = \lambda T_Y(f)$ ✓

(b) let $Y = \{0, \pi/2, \pi\} \subseteq \mathbb{R}$. Then $T_Y(\{f_0, f_1, f_2\}) = \underbrace{\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix}\right\}}$

Claim: is lin. indep.

proof: $\begin{pmatrix}1&1&1\\1&0&-1\\1&-1&1\end{pmatrix} \rightsquigarrow \begin{pmatrix}1&1&1\\0&-1&-2\\0&-2&0\end{pmatrix} \rightsquigarrow \begin{pmatrix}1&1&1\\0&1&0\\0&0&1\end{pmatrix}$ so Col is 3 dim'l QED.

So $T_Y(\{f_0, f_1, f_2\})$ is lin indep & T_Y is linear, so $\{f_0, f_1, f_2\}$ is lin indep QED.

(4)(a) ~~$\frac{d}{dx}(0)=0$~~ $\frac{d}{dx}(1) = 0$ so $\ker(\frac{d}{dx}) \neq 0$ so $\boxed{\text{not injective}}$

Claim: it's $\boxed{\text{surjective}}$ as long as $\forall n \in \mathbb{Z}_{\geq 1}$, $n \neq 0$ in F (if $\exists n \in \mathbb{Z}_{\geq 1}$, $n = 0$ in F then $x^{n-1} \notin \operatorname{im} \frac{d}{dx}$. ex: $F = \mathbb{F}_2$, then $x \neq \frac{d}{dx}(f(x))$)

proof: $\frac{d}{dx}\left(\sum_{i=0}^{\infty} a_i \frac{x^{i+1}}{i+1}\right) = \sum_{i=0}^{\infty} a_i x^i$ QED

(b) It is geometrically clear that $R_{-\theta}\circ R_\theta = id = R_\theta \circ R_{-\theta}$. so R_θ is invertible hence $\boxed{\text{bijective}}$

(c) $\ker(T) = Nul(A)$:
$\dim(T) = Col(A)$
$\begin{pmatrix}1&2&3\\3&5&5\\3&7&8\end{pmatrix} \rightsquigarrow \begin{pmatrix}1&2&3\\0&1&-1\\0&1&-1\end{pmatrix} \rightarrow \begin{pmatrix}1&2&3\\0&1&-1\\0&0&0\end{pmatrix}$ ~~so dim Col~~ so $\dim(\operatorname{im}(T)) = 2 < 3$ & $\dim(\ker(T)) = 1 > 0$

so $\boxed{\text{neither inj. nor surj.}}$

(d) $\begin{pmatrix}1&2&3\\3&5&5\\3&5&8\end{pmatrix} \rightarrow \begin{pmatrix}1&2&3\\0&1&-1\\0&-1&-1\end{pmatrix} \rightsquigarrow \begin{pmatrix}1&2&3\\0&1&-1\\0&0&1\end{pmatrix}$ so $\dim(\operatorname{im}(T)) = 3 = \dim \mathbb{R}^3$, $\dim(\ker(T)) = 0$ so $\boxed{\text{bijective}}$

(4)(e) $\int_0^1 2x\,dx = x^2\Big|_0^1 = 1$ & $\int_0^1 1\,dx = x\Big|_0^1 = 1$

so $I(2x) = I(1) = 1$ so $\boxed{\text{not inj.}}$

(Alternately: $\int_0^1 (x-\frac{1}{2})\,dx = \frac{x^2}{2} - \frac{1}{2}x\Big|_0^1 = \frac{1}{2} - \frac{1}{2} + 0 - 0 = 0$

so $\ker(I) \neq 0$)

$\int_0^1 \alpha\,dx = \alpha\int_0^1 1\,dx = \alpha \quad \forall \alpha \in \mathbb{R}$, so $\boxed{\text{surj.}}$

(5)(a) $\begin{pmatrix}1\\0\\1\end{pmatrix} + \begin{pmatrix}2\\1\\-2\end{pmatrix} + \begin{pmatrix}3\\-1\\3\end{pmatrix} = \begin{pmatrix}6\\0\\2\end{pmatrix}$ so if T is linear, then

$$T\begin{pmatrix}1\\0\\1\end{pmatrix} + T\begin{pmatrix}2\\1\\-2\end{pmatrix} + T\begin{pmatrix}3\\-1\\3\end{pmatrix} = T\begin{pmatrix}6\\0\\2\end{pmatrix}$$

$$\begin{array}{ccc} \| & & \| \\ x+2+x^2-1+2x & & x^2+1 \\ \| & & \| \\ x^2+3x+1 & \neq & x^2+1 \end{array}$$

so $\boxed{\text{no such } T}$

(b) Since $\begin{pmatrix}1\\0\\1\end{pmatrix}$ & $\begin{pmatrix}6\\0\\2\end{pmatrix}$ are lin. indep, yes!

(ex: $\left\{\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}6\\0\\2\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$ is a basis of $\mathbb{R}^3$, so $\exists\, T: \mathbb{R}^3 \to \mathbb{R}[x]$ s.t.

$T\begin{pmatrix}1\\0\\1\end{pmatrix} = x+2$, $T\begin{pmatrix}6\\0\\2\end{pmatrix} = x^2+1$ & $T\begin{pmatrix}0\\1\\0\end{pmatrix} = 0$ (or anything else you want)

(6)(a) same dim, so $\boxed{\text{isom}}$. Let $T: F^4 \to M_{2,2}(F)$ given by $\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \mapsto \begin{pmatrix}1&0\\0&0\end{pmatrix}$

$\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \mapsto \begin{pmatrix}0&1\\0&0\end{pmatrix}$, $\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \mapsto \begin{pmatrix}0&0\\1&0\end{pmatrix}$ & $\begin{pmatrix}0\\0\\0\\1\end{pmatrix} \mapsto \begin{pmatrix}0&0\\0&1\end{pmatrix}$

T is linear. Claim: T is ~~injective~~ surjective

proof: $\begin{pmatrix}a&b\\c&d\end{pmatrix} = T\begin{pmatrix}a\\b\\c\\d\end{pmatrix}$ ✓ QED

Use Theorem: if $T: V \to W$ is linear & $\dim(V) = \dim(W) = n < \infty$ then

T is inj. iff T is surj iff T is bijective

proof: since $\dim(\ker(T)) + \dim(\operatorname{im}(T)) = n$, $\dim \operatorname{im}(T) = n$ iff $\dim \ker(T) = 0$

so injective is equiv. to surj. And bij. is inj. & surj. QED

Thus T is inj. QED

(6)(b) If $\exists n \in \mathbb{Z}_{\geq 1}$, s.t. $n=0$ in F, then by question (4)(a), $\frac{d}{dx}$ is surjective. so $\text{im}\left(\frac{d}{dx}\right) = F[x]$

$F[x] \cong F[x]$ by the identity map.

If $\exists n \in \mathbb{Z}_{\geq 1}$ s.t. $n=0$ in F, then let p be the least element of $\mathbb{Z}_{\geq 1}$ that is zero in F.

Claim: Then $\text{im}\left(\frac{d}{dx}\right) = \text{Span}(\{x^i : i \in \mathbb{Z}_{\geq 0},\ i \neq pk-1 \text{ for some } k\})$

proof:

Claim: p is a prime nb

proof: if p is not prime then $\exists a,b \in \mathbb{Z}$ $p=ab$, $1<a,b<p$. & $ab=0$ in F

so $a^{-1}(ab)=0$ so $b=0$ contrad.

QED.

Claim: $n \in \mathbb{Z}_{\geq 1}$ is 0 in F iff $n=pk$ for some k.

proof: $(\Leftarrow)$ if $n=pk$, then $n=0 \cdot k=0$

$(\Rightarrow)$ if $n \neq pk$, then $\exists\ 0<r<p$ st. $n=pk+r$ for some k

so $n=0\cdot k+r=r<p$ so $r \neq 0$, since p is least. QED

So if $i \neq pk-1$, then $i+1 \neq 0$ in F so $\frac{1}{i+1} \in F$ so $\frac{x^{i+1}}{i+1} \in F[x]$

so $\frac{d}{dx}\left(\frac{x^{i+1}}{i+1}\right) = x^i \in F[x]$

so $\text{im}\left(\frac{d}{dx}\right) \supseteq \text{Span}\{x^i : i \in \mathbb{Z}_{\geq 0},\ i \neq pk-1\}$

claim: If $f(x) = \sum_{k=1}^{\infty} a_k x^{pk-1}$, then $f \notin \text{im}\left(\frac{d}{dx}\right)$

prof: since $\frac{d}{dx}(x^n) \in \text{Span}(x^{n-1})$

any $g(x)$ st. $\frac{d}{dx}g(x) = f(x)$ must be $\in \text{Span}\{x^{pk} : k \in \mathbb{Z}_{\geq 1}\}$

but $\frac{d}{dx}\left(\text{span}(\{x^{pk} : k \in \mathbb{Z}_{\geq 1}\})\right) = 0$ QED.

Claim: If $f(x) = \sum_{j=0}^{\infty} a_j x^j$ & $\exists j_0$ st. $a_{j_0} \neq 0$ & $j=pk-1$, then $f \notin \text{im}\left(\frac{d}{dx}\right)$

prof: let $f_1(x) = \sum_{\substack{j=0 \\ j \neq pk-1}} a_j x^j$ then $f_1 \in \text{im}\left(\frac{d}{dx}\right)$

if $f \in \text{im}\left(\frac{d}{dx}\right)$, so is $f - f_1 = \sum_{k=1} b_k x^{pk-1}$. contrad. QED

QED.

so $\text{im}\left(\frac{d}{dx}\right) = \text{Span}\{x^i : i \neq pk-1\}$. Then $F[x] \cong \text{im}\left(\frac{d}{dx}\right)$ via sending

$1 \mapsto 1$

$x \mapsto x$

$\vdots$

$x^{p-2} \mapsto x^{p-2}$

$x^{p-1} \mapsto x^p$

$\vdots$

$x^{2p-1} \mapsto x^{2p}$ etc.

QED

(7)(a) $T\begin{pmatrix}1\\0\\0\\0\end{pmatrix}=\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$, $T\begin{pmatrix}0\\1\\0\\0\end{pmatrix}=\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$, $T\begin{pmatrix}0\\0\\1\\0\end{pmatrix}=\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}$, $T\begin{pmatrix}0\\0\\0\\1\end{pmatrix}=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$

so $\boxed{{}_B[T]_B=\begin{pmatrix}1&1&1&1\\1&0&1&0\\1&0&-1&1\\1&1&-1&2\end{pmatrix}}$

(b) We want to write $T(e_i)$ as $x_1\overbrace{(e_1+e_2)}^{v_1}+x_2\overbrace{(e_2+e_3)}^{v_2}+x_3\overbrace{(e_3+e_4)}^{v_3}+x_4\overbrace{e_4}^{v_4}$

so solve $\left(\begin{array}{cccc|cccc}1&0&0&0&1&1&1&1\\1&1&0&0&1&0&1&0\\0&1&1&0&1&0&-1&1\\0&0&1&1&1&1&-1&2\end{array}\right)$

well $\begin{pmatrix}1\\1\\1\\1\end{pmatrix}=v_1+v_3$, $\begin{pmatrix}1\\0\\0\\1\end{pmatrix}=v_1-v_2+v_3$ $\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}=v_1-v_3$ & $\begin{pmatrix}1\\0\\1\\2\end{pmatrix}=v_1-v_2+2v_3$

So $\boxed{{}_C[T]_B=\begin{pmatrix}1&1&1&1\\0&-1&0&-1\\1&1&-1&2\\0&0&0&0\end{pmatrix}}$

(c) $T\begin{pmatrix}1\\1\\0\\0\end{pmatrix}=\begin{pmatrix}2\\1\\1\\2\end{pmatrix}$, $T\begin{pmatrix}0\\1\\1\\0\end{pmatrix}=\begin{pmatrix}2\\1\\-1\\0\end{pmatrix}$, $T\begin{pmatrix}0\\0\\1\\1\end{pmatrix}=\begin{pmatrix}2\\1\\0\\1\end{pmatrix}$, $T\begin{pmatrix}0\\0\\0\\1\end{pmatrix}=\begin{pmatrix}1\\0\\1\\2\end{pmatrix}$

so $\boxed{{}_B[T]_C=\begin{pmatrix}2&2&2&1\\1&1&1&0\\1&-1&0&1\\2&0&1&2\end{pmatrix}}$

(d) look at ${}_C[T]_B=\begin{pmatrix}1&1&1&1\\0&-1&0&-1\\1&1&-1&2\\0&0&0&0\end{pmatrix}\rightsquigarrow\begin{pmatrix}1&1&1&1\\0&1&0&1\\0&0&-2&1\\0&0&0&0\end{pmatrix}\to\begin{pmatrix}1&0&0&1\\0&1&0&1\\0&0&1&-1/2\\0&0&0&0\end{pmatrix}$

row of zeros

so rk ≤ 3 so $\boxed{\text{not surj}}$

so null $= 4-\text{rk}\geq 1$ $\boxed{\text{so not inj.}}$

$\boxed{\text{so basis of image is } \{v_1+v_3,\ v_1-v_2+v_3,\ v_1-v_3\}}$

$\boxed{\text{basis of kernel is } \{-v_1-v_2+\tfrac{1}{2}v_3+v_4\}}$

(8)(a) $\int_0^x t^i\,dt = \frac{x^{i+1}}{i+1}$ for $i=0,\dots,d$

so ${}_{\mathcal{E}}[I]_{\mathcal{B}} = \begin{pmatrix} 0 & 0 & 0 & \cdot & & \\ 1 & 0 & 0 & & & \\ 0 & 2^{-1} & 0 & & & \vdots \\ 0 & 0 & 3^{-1} & & & \\ 0 & 0 & 0 & \ddots & & 0 \\ \vdots & \vdots & \vdots & & & 0 \\ 0 & \cdots & & & 0 & (d+1)^{-1} \end{pmatrix}$

(b) Placing the top row at the bottom, we see that there are $d+1$ pivots, so

$\dim \operatorname{Im}(I) = d+1 < d+2 = \dim \mathbb{R}[x]_{\le d+1}$, so not <u>surjective.</u>

$\qquad\qquad\quad \|$

$\dim \mathbb{R}[x]_{\le d}$ so <u>injective</u> so $\ker(I)=0$, so <u>no basis</u>

Image is $\boxed{\text{Span}\{x^i : 1 \le i \le d+1\} \text{ which is a basis}}$ (The matrix says that the coord vectors wrt $\mathcal{E}$

$\begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2^{-1} \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 3^{-1} \\ 0 \\ \vdots \end{pmatrix}, \dots$ are a basis

These correspond to $\frac{x^{i+1}}{i+1}$ $i=0,\dots,d$)

(c) $I\left(\sum_{i=0}^{k} x^i\right) = \sum_{i=0}^{k} \frac{x^{i+1}}{i+1}$

so ${}_{\mathcal{E}}[I]_{\mathcal{B}'} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 2^{-1} & 2^{-1} & 2^{-1} & 2^{-1} & \cdots & 2^{-1} \\ 0 & 0 & 3^{-1} & 3^{-1} & & & 3^{-1} \\ 0 & 0 & 0 & 4^{-1} & & & 4^{-1} \\ 0 & 0 & 0 & 0 & & & \vdots \\ & & & & & & (d+1)^{-1} \end{pmatrix}$

(d) $I(x^i) = \frac{x^{i+1}}{i+1} = \frac{1}{i+1}\left(\sum_{j=0}^{i+1} x^j - \sum_{j=0}^{i} x^j\right)$

so ${}_{\mathcal{E}}[I]_{\mathcal{B}'} = \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 \\ 1 & -2^{-1} & 0 & \cdots & 0 \\ 0 & 2^{-1} & -3^{-1} & \ddots & 0 \\ 0 & 0 & 3^{-1} & \ddots & \vdots \\ 0 & 0 & 0 & & \vdots \\ & & \ddots & & -(d+1)^{-1} \\ & & & \ddots & (d+1)^{-1} \end{pmatrix}$