Math 411
Asst. 8 solutions

(1) (a) $\forall v \in V$, $T_0(v) = 0 \cdot v = 0$ so $T_0(v) = 0$ $\forall v$. so $T_0 = 0$

(b) $T_1(v) = 1 \cdot v = v$ so $\forall v$, $T_1(v) = v$ so $T_1 = id = 1$

(c) $T_{\alpha_1+\alpha_2}(v) = (\alpha_1+\alpha_2)\cdot v = \alpha_1 \cdot v + \alpha_2 \cdot v = T_{\alpha_1}(v) + T_{\alpha_2}(v) = (T_{\alpha_1} + T_{\alpha_2})(v)$

(d) $T_{\alpha_1\alpha_2}(v) = (\alpha_1\alpha_2)\cdot v = \alpha_1 \cdot (\alpha_2 v) = \alpha_1 \cdot (T_{\alpha_2}(v)) = T_{\alpha_1}(T_{\alpha_2}(v)) = (T_{\alpha_1} \circ T_{\alpha_2})(v) = (T_{\alpha_1} T_{\alpha_2})(v)$

(2) (a) Assume T is surjective. Let $\mathcal{E}$ be a basis of W. Since T is surj, $\forall c \in \mathcal{E}$, $\exists v_c \in V$ st. $T(v_c) = c$.
Since $\dim(W) > \dim(V)$, the set $\{v_c : c \in \mathcal{E}\}$ contains more vectors than any basis of V & hence must be linearly dep. So $\exists \alpha_c \in F$ not all zero s.t.

$$\sum_{c\in\mathcal{E}} \alpha_c v_c = 0. \text{ Then } T\Big(\sum_{c\in\mathcal{E}} \alpha_c v_c\Big) = 0$$
$$\qquad\qquad\qquad\qquad \| $$
$$\sum_{c\in\mathcal{E}} \alpha_c c$$

. So $\mathcal{E}$ is lin. dep.: contradiction!

(b) Assume T is inj. Let B be a basis of V. Since T is injective $T(B)$ is lin. indep.; but $\# T(B) >$ size of any basis of W, so $T(B)$ must be lin. dep. contrad!

(3) (a)
$$\int_{-1}^{1} 1\,dx = 2$$
$$\int_{-1}^{1} x\,dx = 0$$
$$\int_{-1}^{1} x^2\,dx = \frac{x^2}{2}\Big|_{-1}^{1} = 1$$
$$\int_{-1}^{1} \frac{3x^2-1}{2}\,dx = \frac{x^3}{2} - \frac{x}{2}\Big|_{-1}^{1} = \frac{1}{2} - \frac{1}{2} - \left(\frac{-1}{2} + \frac{1}{2}\right) = 0$$

$$\boxed{{}_B[I]_B = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2/5 & 0 \\ 0 & 0 & 0 & 2/7 \end{pmatrix}}$$

$$\boxed{\int_{-1}^{1} x\left(\frac{5x^3-3x}{2}\right)dx = \frac{2}{2}\cdot\int_0^1 (5x^4 - 3x^2)\,dx = 1 - 1 = 0}$$

$$\int_{-1}^{1} x \cdot \frac{(3x^2-1)}{2}\,dx = \int_{-1}^{1} \left(\frac{3x^3}{2} - \frac{x}{2}\right)dx = \frac{3x^4}{8} - \frac{x^2}{4}\Big|_{-1}^{1} = \frac{3}{8} - \frac{1}{4} - \left(\frac{3}{8} - \frac{1}{4}\right) = 0$$

$$\int_{-1}^{1} \left(\frac{3x^2-1}{2}\right)^2 dx = \frac{1}{4}\int_{-1}^{1} (9x^4 - 6x^2 + 1)\,dx = \frac{1}{4}\left(\frac{9x^5}{5} - 2x^3 + x\Big|_{-1}^{1}\right) = \frac{1}{4}\left(\frac{9}{5} - 2 + 1 - \left(\frac{-9}{5} + 2 - 1\right)\right)$$
$$= \frac{1}{2}\left(\frac{9}{5} - \frac{10}{5} + \frac{5}{5}\right) = \frac{2}{5}$$

$\int_{-1}^{1} \left(\frac{5x^3-3x}{2}\right) dx = 0$ (its odd). $\int_{-1}^{1} \left(\frac{5x^3-3x}{2}\right)\cdot\left(\frac{3x^2-1}{2}\right) dx = 0$ (its odd)

$$\int_{-1}^{1} \left(\frac{5x^3-3x}{2}\right)^2 = \frac{2}{4}\int_0^1 (25x^6 - 30x^4 + 9x^2)\,dx = \frac{1}{2}\left(\frac{25}{7} - \frac{30}{5} + \frac{9}{3}\right) = \frac{1}{2}\left(\frac{25-42+21}{7}\right) = \boxed{\frac{2}{7}}$$

(3) (b) let $B' = \{1, x, x^2, x^3\}$

$$\int_{-1}^{1} 1\, dx = 2$$

$$\int_{-1}^{1} x\, dx = \int_{-1}^{1} x^3\, dx = \int_{-1}^{1} x^5\, dx = 0$$

(odd functions)

$$\int_{-1}^{1} x^2\, dx = \frac{x^3}{3}\Big|_{-1}^{1} = \frac{2}{3}$$

$$\int_{-1}^{1} x^4\, dx = 2\int_{0}^{1} x^4\, dx = \frac{2}{5}$$

$$\int_{-1}^{1} x^6\, dx = 2\int_{0}^{1} x^6\, dx = \frac{2}{7}$$

$$\boxed{{}_{B'}[I]_{B'} = \begin{pmatrix} 2 & 0 & 2/3 & 0 \\ 0 & 2/3 & 0 & 2/5 \\ 2/3 & 0 & 2/5 & 0 \\ 0 & 2/5 & 0 & 2/7 \end{pmatrix}}$$

(4) (a) $I(\alpha f_1(x) + f_2(x), g(x)) = \int_a^b (\alpha f_1(x) + f_2(x))\, g(x)\, w(x)\, dx$

$$= \int_a^b (\alpha f_1(x) g(x) w(x) + f_2(x) g(x) w(x))\, dx$$

$$= \alpha \int_a^b f_1(x) g(x) w(x)\, dx + \int_a^b f_2(x) g(x) w(x)\, dx = \alpha I(f_1(x), g(x)) + I(f_2, g)$$

similarly for $I(f(x), \alpha g_1(x) + g_2(x))$.

(b) Let $B = \{f_0, f_1, f_2\} = \{1, \cos(x), \cos(2x)\}$

$$\boxed{{}_{B}[I]_{B} = \begin{pmatrix} \pi & 0 & 0 \\ 0 & \pi/2 & 0 \\ 0 & 0 & \pi/2 \end{pmatrix}}$$

$$\int_0^\pi 1\, dx = \pi$$

$$\int_0^\pi \cos(x)\, dx = \sin(x)\Big|_0^\pi = 0$$

$$\int_0^\pi \cos^2(x)\, dx = \int_0^\pi \frac{(1 + \cos(2x))}{2}\, dx = \int_0^\pi \frac{1}{2}\, dx + \frac{1}{2}\int_0^\pi \cos(2x)\, dx = \frac{\pi}{2}$$

$$\int_0^\pi \cos(2x)\, dx = \frac{\sin(2x)}{2}\Big|_0^\pi = 0$$

$$\int_0^\pi \cos^2(2x)\, dx = \int_0^\pi \frac{1}{2}\, dx + \int_0^\pi \frac{\cos(4x)}{2}\, dx = \frac{\pi}{2} + \frac{\sin(4x)}{8}\Big|_0^\pi = \pi/2$$

$$\int_0^\pi \cos(x)\cos(2x)\, dx = \int_0^\pi \cos(x)(\cos^2(x) - \sin^2(x))\, dx$$

$$= \int_0^\pi \cos(x)(1 - \sin^2(x) - \sin^2(x))\, dx = \overbrace{\int_0^\pi \cos(x)\, dx}^{0} - 2\int_0^\pi \cos(x)\sin^2(x)\, dx$$

$u = \sin(x)$

$du = \cos(x)\, dx$

$$= -2\int_0^0 \sim = 0$$