Asst. 9 Solutions
Math 411

(1) $A=\begin{pmatrix}1&4\\2&5\\3&6\end{pmatrix}$: It cannot be non-degenerate because the map $v\mapsto B_v$ from F^3 to $(F^2)^*$ cannot be injective because it is a map from a 3-dim'l to a 2 dim'l space.

Note however that the map $w\mapsto B_w$ from F^2 to $(F^3)^*$ is injective:

proof: recall from Q(2) below that the 1×3 matrix representing B_w is $(Aw)^T$

so $Be_1 = (1\ 0)\begin{pmatrix}1&2&3\\4&5&6\end{pmatrix} = (1\ 2\ 3)$

& $Be_2 = (0\ 1)\begin{pmatrix}1&2&3\\4&5&6\end{pmatrix} = (4\ 5\ 6)$

so the matrix of $w\mapsto B_w$ in the standard bases of F^2 & $(F^3)^*$ is

$A=\begin{pmatrix}1&4\\2&5\\3&6\end{pmatrix}$ ~~$\begin{pmatrix}1&2&3\\4&5&6\end{pmatrix}$~~ which row reduces to $\begin{pmatrix}1&4\\0&-3\\0&-6\end{pmatrix}$ which has no free variables so $\ker(w\mapsto B_w)=0$. QED

$A=\begin{pmatrix}1&2\\3&6\\4&8\end{pmatrix}$: Like above it cannot be non-degen. since $v\mapsto B_v$ can't be injective but here $w\mapsto B_w$ is also not injective.

Indeed, as above its matrix is A, which now row reduces to

$\begin{pmatrix}1&2\\0&0\\0&0\end{pmatrix}$, so $\ker(w\mapsto B_w)\neq 0$ is 1 dim'l.

(2) (a) v is $m\times 1$ & A is $m\times n$ so v^TA is $1\times n$ & hence represents a linear functional $F^n\to F$.

so $B_v(w)=(v^TA)w$ for all $w\in F^n$, it represents B_v. QED

(b) similarly $(Aw)^T$ is $1\times m$. Now $B_w(v)=v^TAw$. This is a 1×1 matrix

& so $=(v^TAw)^T = (Aw)^Tv$. QED

(c) The only way maps from $F^n\to(F^m)^*$ & $F^m\to(F^n)^*$ can be injective is if $n\le m$ & $m\le n$, respectively (because of dimension reasons). So if B is non-degen. then $m=n$.

(d) ($\Rightarrow$) If B is non-degenerate, then the map $w\mapsto B_w$ is injective. Since B_w is represented by $(Aw)^T$, the injectivity means that $(Aw)^T=0$ only for $w=0$. I.e. 0 is the only sol'n to $Ax=0$, i.e. $\mathrm{Nul}(A)=0$.

(2) (d) (cont'd) ($\Leftarrow$) Suppose B is degenerate, so that $w \mapsto B_w$ or $v \mapsto B_v$ is not injective. If $w \mapsto B_w$ is not injective, then $\exists w \neq 0$ s.t. $B_w = 0$, ie. s.t. $(Aw)^T = 0$, ie. s.t. $Aw = 0$. so $Nul(A) \neq 0$.

If $v \mapsto B_v$ is not injective, then $\exists v \neq 0$ s.t. $B_v = 0$, ie. s.t. $v^T A = 0$, ie. s.t. $A^T v = 0$ so $Nul(A^T) \neq 0$. Since $m = n$ this means $Nul(A) \neq 0$ (since $rk(A^T) = \text{rowrk}(A) = rk(A)$)

QED

(3)(a) B nondegen $\Rightarrow$ V injects into W^* & W injects into V^*. $\dim V < \infty$ & $\dim W < \infty$ so this means $W \cong W^*$ & $V \cong V^*$ so $\dim V \leq \dim W$ & $\dim W \leq \dim V$ so $\dim V = \dim W$.

$\dim V^* = \dim W^*$

An injective map of finite-dim'l spaces of the same dim is necessarily an isomorphism so $V \cong W^*$ & $W \cong V^*$.

(b) Let $\mathcal{C}, \mathcal{C}'$ bases of V & W, $\mathcal{C} = \{c_1, \ldots, c_n\}$, $\mathcal{C}' = \{c_1', \ldots, c_n'\}$ & ~~let a_{ij}~~ let $a_{ij} \in F$

let $A = (a_{ij}) \in M_n(F)$ & define $B(v,w) = [v]_{\mathcal{C}}^T A [w]_{\mathcal{C}'}$

Then B is a bilinear form $V \times W \to F$

(Indeed, for fixed v, the map $w \mapsto B(v,w)$ is the composition of the isomorphism $W \xrightarrow{\sim} F^n$ with the map mult. by $([v]_{\mathcal{C}}^T A)$ $F^n \to F$; similarly for $v \mapsto B(v,w)$).

by Q (2) part (d), this will be non-degen. iff $Nul(A) = 0$.

For $A = (\delta_{ij})$ = identity matrix, clearly $Nul(A) = 0$ QED

(c) We have to show that $B_{c_i}(c_j) = \delta_{ij}$, since then $c_i^* = B_{c_i}$.

Well, $B_{c_i}(c_j) = B(c_i, c_j) = \delta_{ij}$ by definition QED

(4)(a) We know from class that B is bilinear, & symmetric (since M is symmetric).

We must show B is positive definite. Well, let $v = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$, let $B\left(\begin{pmatrix} a \\ b \\ c \end{pmatrix}, \begin{pmatrix} a \\ b \\ c \end{pmatrix}\right) =$

$(a\ b\ c)\begin{pmatrix} 1 & & \\ & 2 & \\ & & 3 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = a^2 + 2b^2 + 3c^2$. Since squares are ≥ 0, $B(v,v) \geq 0$ $\forall v \in \mathbb{R}^3$.

& if $x \neq 0$, then $x^2 > 0$, so $a^2 + 2b^2 + 3c^2 = 0$ iff $a = b = c = 0$. QED.

(b) $B(Av, w) = (Av)^T M w = v^T A^T M w = v^T((A^T M) w)$

We want a matrix A^* s.t. $v^T((A^T M) w) = v^T(M(A^* w))$

i.e. $A^T M = M A^*$, so $A^* = M^{-1} A^T M$

$$= \begin{pmatrix} 1 & & \\ & 1/2 & \\ & & 1/3 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 2 & \\ & & 3 \end{pmatrix}}_{\shortparallel}$$

$$= \begin{pmatrix} 1 & & \\ & 1/2 & \\ & & 1/3 \end{pmatrix} \begin{pmatrix} 1 & 8 & 21 \\ 2 & 10 & 34 \\ 3 & 12 & 27 \end{pmatrix} = \begin{pmatrix} 1 & 8 & 21 \\ 1 & 5 & 17 \\ 1 & 4 & 9 \end{pmatrix}$$

so $T^*(v) = A^* v$ $\quad A^* =$ (the matrix above)

(5) $\Leftarrow$ Suppose $\lambda_i > 0\ \forall i$. let $v = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$, then $B(v,v) = (a_1 \cdots a_n) \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$

$= \lambda_1 a_1^2 + \cdots + \lambda_n a_n^2 \geq 0$

& $= 0$ iff $a_i = 0\ \forall i$ ✓

$\Rightarrow$ if $\exists \lambda_i \leq 0$, let $v = \begin{array}{r} \\ \\ \text{ith entry} \rightarrow \\ \\ \\ \end{array} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, then $B(v,v) = \lambda_i \leq 0$. But $v \neq 0$.

so B is not pos. def. ✓

QED

(6) (a) We've already seen this is bilinear & it's clearly symmetric (since $f(x)g(x) = g(x)f(x)$) so show that it's pos. def.

let $f(x) \in \mathbb{R}[x]$, then $\langle f(x), f(x) \rangle = \int_0^1 f(x)^2\, dx$. since $f(x)^2 \geq 0$, its integral ≥ 0, & $= 0$ iff $f(x) = 0$ (since $f(x)$ is cont. ~~& has finitely many zeroes in [0,1]~~)

& has finitely many zeroes, if $f(x) \neq 0$

precisely, ~~a cont. function on a closed interval has a minimum value~~ if $f(x) \neq 0$.

since ~~$f(x)^2 \geq 0$~~ $\exists x_0 \in [0,1)$ s.t. $f(x_0) \neq 0$, there is some interval containing x_0 inside $[0,1)$ where $f(x) \neq 0$ $\forall x$ in the interval $[x_0 - \varepsilon, x_0 + \varepsilon]$

~~Plea~~ f is cont. so $\exists m > 0$ s.t. $|f(x)| \geq m$ for $x_0 - \varepsilon < x < x_0 + \varepsilon$

so $\int_0^1 f(x)\, dx \geq 2\varepsilon \cdot m > 0$.

(6)(b) Claim: $\langle f(x), xg(x)\rangle = \langle xf(x), g(x)\rangle$

$$\int_0^1 f(x)\,xg(x)\,dx = \int_0^1 x f(x) g(x)\,dx \quad \text{QED}$$

(c) $\langle \alpha f_1(x) + f_2(x), g(x)\rangle_{\ell^2} = \sum_{n=0}^{\infty} (\alpha a_{1,n} + a_{2,n}) b_n = \alpha \sum_{n=0}^{\infty} a_{1,n} b_n + \sum_{n=0}^{\infty} a_{2,n} b_n$

$= \alpha \langle f_1, g\rangle + \langle f_2, g\rangle$

Clearly $\langle \cdot,\cdot\rangle_{\ell^2}$ is symmetric, so this shows it's linear in $g(x)$, too.

pos. def: $\langle f(x), f(x)\rangle_{\ell^2} = \sum_{n=0}^{\infty} a_n^2 \geq 0$ & $=0$ iff $a_n = 0\ \forall n$.

(d) let $f(x) = x^2$, $g(x) = x^3$, then $\langle x f(x), g(x)\rangle_{\ell^2} = \langle x^3, x^3\rangle_{\ell^2} = 1$

but $\langle f(x), xg(x)\rangle_{\ell^2} = \langle x^2, x^4\rangle_{\ell^2} = 0$

so T is not self-adjoint.

(e) Claim: $\langle Tf, g\rangle_{\ell^2} = \langle f, Lg\rangle_{\ell^2}$ $\forall f, g$.

proof: it's true as long as it's true on a basis. so let's show it's true for $f(x) = x^i$, $g(x) = x^j$

$$\langle Tx^i, x^j\rangle = \langle x^{i+1}, x^j\rangle = \delta_{i+1,j}$$

$$\langle x^i, Lx^j\rangle = \langle x^i, x^{j-1}\rangle = \delta_{i,j-1}$$

but $i+1 = j$ iff $i = j-1$

so $\delta_{i+1,j} = \delta_{i,j-1}$ QED