Assignment 11 – All 2 parts – Math 412

Due: Tuesday, Nov. 29, 2016, at the beginning of class

Textbook exercises:[1]

Section 5.3: 8 (WI), 11

Other exercises:

(1) Recall that in class we considered the field $\mathbf{F}_4 = (\mathbf{Z}/2\mathbf{Z})[x]/(x^2 + x + 1)$. Let $f_4(x) = x^4 - x \in \mathbf{F}_4[x]$. Show that every element of $\mathbf{F}_4$ is a root of $f_4(x)$. (Remark: and so, since a polynomial of degree n has at most n roots in a field, $\mathbf{F}_4$ is exactly the set of all roots of $f_4(x)$ in itself.)

(2) Write out the multiplication table for the ring $(\mathbf{Z}/2\mathbf{Z})[x]/(x^3 + x + 1)$. Is this ring a field?

(3) Write out the multiplication table for the ring $(\mathbf{Z}/2\mathbf{Z})[x]/(x^3 + x^2 + x + 1)$. Is this ring a field?

(4) Let p be a prime number. Recall from class that the pth cyclotomic polynomial is

$$\Phi_p(x) = x^{p-1} + x^{p-2} + \cdots + x + 1$$

is an irreducible polynomial in $\mathbf{Q}[x]$ and is such that

$$x^p - 1 = (x - 1) \cdot \Phi_p(x).$$

Let $K = \mathbf{Q}(\zeta_p)$ be the field obtained by adjoining a root ζ_p of $\Phi_p(x)$ to $\mathbf{Q}$. Show that the inverse of ζ_p is $(\zeta_p)^{p-1}$.

(5) Let $n(x) = x^3 - 2 \in \mathbf{Q}[x]$.

(a) Why is $n(x)$ irreducible?

(b) Let $K = \mathbf{Q}[x]/(n(x))$, so that $K = \mathbf{Q}(\alpha)$, where α is a root of $n(x)$ (you may as well think of α as $\sqrt[3]{2}$). Find the polynomial expression $a_0 + a_1\alpha + a_2\alpha^2 \in K$ that is the inverse of $\alpha - 1$. (By the way, this expression has $a_0, a_1, a_2 \in \mathbf{Z}$.)

[1] From Hungerford's *Abstract algebra, An introduction, Third edition*