Assignment 12 – The only part – Math 412

Due: Thursday, Dec. 1, 2016, at the beginning of class

Textbook exercises:[1]

Section 6.1: 2 (WI)

Other exercises:

(1) (WI) Let R be a commutative ring and fix $a \in R$. Let $I = \{f(x) \in R[x] : f(a) = 0\}$. Show that I is an ideal of $R[x]$.

(2) Let R be the ring of functions $f : \mathbf{R} \to \mathbf{R}$ and let U be any subset of $\mathbf{R}$. Show that $\{f : \mathbf{R} \to \mathbf{R} : f(u) = 0 \text{ for all } u \in U\}$ is an ideal of R.

(3) Let R be a ring and let A be a non-empty subset of R. We defined (A) to be the smallest ideal of R containing A.

   (a) In order to know such a 'smallest' ideal always exists, we need to know that intersections of ideals are ideals. Show that if X is a non-empty set and I_x is an ideal of R for every $x \in X$, then $\bigcap_{x \in X} I_x$ is an ideal of R.

   (b) However, the union of two ideals may not be an ideal. Show this is true by showing that $(2) \cup (3)$ is not an ideal in $\mathbf{Z}$.

   (c) Given two ideals I, J of R, their sum $I + J$ is defined to be

$$I + J = \{a + b : a \in I, b \in J\}.$$

   Show that $I + J$ is an ideal and that it is the smallest ideal containing $I \cup J$.

(4) Let R be any ring.

   (a) Show that $\{0\} \subseteq R$ is an ideal of R. It is called the zero ideal and usually just denoted 0.

   (b) By the way, the ring R itself is an ideal in R. Show that if I is an ideal of R and I contains a unit, then $I = R$. For this reason, the ideal R is called the *unit ideal*.

[1]From Hungerford's *Abstract algebra, An introduction, Third edition* TURN OVER

(c) A ring R is called *simple* if $R \neq 0$ and its only ideals are 0 and R itself. Show that every division ring is simple.

(d) Suppose R is a simple commutative ring. Show that R is a field. (Hint: you want to show that if $a \in R$ is non-zero, then there is an $r \in R$ such that $ra = 1$, so consider the principal ideal (a) in R.) (So, people don't usually talk about 'simple-ness' with commutative rings because a simple commutative ring is the same thing as a field.)

(5) Let R be a ring. A subset $I \subseteq R$ is called a *left ideal* if

(i) I is nonempty,

(ii) for all $a, b \in I$, $a - b \in I$,

(iii) for all $a \in I$ and for all $r \in R$, $ra \in I$,

similarly, a *right ideal* is defined like this but (iii) becomes: for all $a \in I$ and for all $r \in R$, $ar \in I$. What we called an ideal is sometimes called a 'two-sided ideal' because it is exactly that same as a subset that is both a left ideal and a right ideal. Note that if R is commutative, then left ideals, right ideals, and two-sided ideals are all the same thing.

(a) Show that if $a \in R$, then the set $Ra = \{ra : r \in R\}$ is a left ideal of R.

(b) Let $R = M_2(\mathbf{R})$ be the ring of 2×2 matrices over $\mathbf{R}$ and let

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in M_2(\mathbf{R}).$$

Show that the left ideal RA is not a right ideal. (Hint: by multiplying an arbitrary matrix on the left of A, you can determine exactly what RA looks like, and then you can multiply on the right to see that not all of those products will be in RA.)