Assignment 3 – All 2 parts – Math 412

Due: Thursday, Sept. 15, 2016, at the beginning of class

Textbook exercises:[1]

Section 2.2: 1(a) and 1(b), 14

Section 2.3: 12 (WI), 13 (WI)

Other exercises:

(1) We saw that $\mathbf{Z}/n\mathbf{Z} = \{[0], [1], \ldots, [n-1]$, but that there can be other ways of writing down representatives of the congruence classes. Here are some more examples of that.

(a) Show that $\mathbf{Z}/5\mathbf{Z} = \{[-2], [-1], [0], [1], [2]\}$ and $\mathbf{Z}/6\mathbf{Z} = \{[-2], [-1], [0], [1], [2], [3]\}$.

(b) Show that (with some terrible notation)

$$\mathbf{Z}/n\mathbf{Z} = \Big\{\big[-\lfloor (n-1)/2 \rfloor\big], \big[-\lfloor n/2 \rfloor + 1\big], \ldots, \big[\lfloor n/2 \rfloor\big]\Big\},$$

where, $\lfloor x \rfloor$ means the 'floor' of x, i.e. the biggest integer m that is $\leq x$.

(c) Show that $\mathbf{Z}/5\mathbf{Z} = \{[0]\} \cup \{[2^0], [2^1], [2^2], [2^3]\}$ and $\{[0]\} \cup \{[3^0], [3^1], [3^2], [3^3]\}$.

(d) Show that $\mathbf{Z}/7\mathbf{Z} = \{[0]\} \cup \{[3^0], [3^1], \ldots, [3^5]\}$, but is not equal to $\{[0]\} \cup \{[2^0], [2^1], \ldots, [2^5]\}$.

(e) Show that there is no $a \in \mathbf{Z}$ such that $\mathbf{Z}/4\mathbf{Z} = \{[0]\} \cup \{[a^0], [a^1], [a^2]\}$.

(2) (WI) We say that the cancellation law holds in $\mathbf{Z}/n\mathbf{Z}$ if for every non-zero $a \in \mathbf{Z}/n\mathbf{Z}$

$$ab = ac \text{ implies } b = c.$$

(a) Show that if $\mathbf{Z}/n\mathbf{Z}$ has a zerodivisor, then the cancellation law does *not* hold in $\mathbf{Z}/n\mathbf{Z}$.

(b) Conversely, show that if $\mathbf{Z}/n\mathbf{Z}$ has no zerodivisors, then the cancellation law does hold. Thus, the cancellation law holding in $\mathbf{Z}/n\mathbf{Z}$ is equivalent to there being no zero divisors in $\mathbf{Z}/n\mathbf{Z}$.

(3) Find all zerodivisors and units in $\mathbf{Z}/n\mathbf{Z}$ for $n = 4, 5, 6, 7$.

[1]From Hungerford's *Abstract algebra, An introduction, Third edition*

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(4) Suppose a is a unit in $\mathbf{Z}/n\mathbf{Z}$ and let $b \in \mathbf{Z}/n\mathbf{Z}$. Show that the equation $ax = b$ has one and only one solution in $\mathbf{Z}/n\mathbf{Z}$.

(5) Consider the set $\mathbf{Z}$ with the following new binary operations $\oplus$ and $\odot$:

$$a \oplus b = a + b - 1 \quad \text{and} \quad a \odot b = a + b - a \cdot b,$$

where $+$ and $\cdot$ denote the usual addition and multiplication of integers. Show that $(\mathbf{Z}, \oplus, \odot)$ is a ring! (Hint: the additive identity is 1 and the multiplicative identity is 0.)