Assignment 4 – All 2 parts – Math 412

Due: Thursday, Sept. 22, 2016, at the beginning of class

Textbook exercises:[1]

Section 3.2: 21, 22, 38(WI)

Other exercises:

(1) An element r of a ring R is called *nilpotent* if one of its powers is 0, i.e. if there is an $n \in \mathbf{Z}_{\geq 1}$ such that $r^n = 0$. (In particular, 0 is always nilpotent by definition.)

(a) Show that $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is nilpotent in the ring $M_2(\mathbf{R})$ of 2×2 matrices with coefficients in $\mathbf{R}$.

(b) Show that $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ is nilpotent in the ring $M_3(\mathbf{R})$.

(c) Show that 6 is a nilpotent element of the ring $\mathbf{Z}/12\mathbf{Z}$.

(d) (WI) Let $n \in \mathbf{Z}_{\geq 2}$. Show that $\mathbf{Z}/n\mathbf{Z}$ has a non-zero nilpotent element if and only if there is a prime p such that p^2 divides n (i.e. n is not squarefree).

(2) Let R be a ring where $0 = 1$. Show that R has only one element. This ring is called the *zero ring* and is denoted by 0.

(3) An element a of a ring R is called a *zerodivisor* if $a \neq 0$ and there is an element $b \in R$ such that $ab = 0$ or $ba = 0$. An element $a \in R$ is called a *unit* if there is a $u \in R$ such that $au = 1 = ua$.

(a) Show that a ring R satisfies the Cancellation Law if and only if it has no zerodivisors.

(b) Show that a unit can't be a zerodivisor.

(4) Let $R = M_2(\mathbf{Z}/2\mathbf{Z})$ be the ring of 2×2 matrices with entries in $\mathbf{Z}/2\mathbf{Z}$.

(a) How many elements are there in R?

(b) Determine the units of R, the zerodivisors of R, and the nilpotent elements of R.

[1]From Hungerford's *Abstract algebra, An introduction, Third edition*

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(5) Let $X = \{x, y\}$ be a set with two elements and let $R = \mathcal{P}(X)$ be the set of subsets of X. Write out the addition and multiplication tables for the ring R where the addition and multiplication are defined as in class to be the symmetric difference and the intersection, respectively:

$$Y + Z = Y \triangle Z = (Y \setminus Z) \cup (Z \setminus Y)$$
$$Y \cdot Z = Y \cap Z.$$

(6) (WI) Let R be a ring. Recall that in class we said that R has characteristic $n \in \mathbf{Z}_{\geq 1}$ if n is the least positive integer such that $n \cdot a = 0$ for all $a \in R$, or if there is no such n, we say the characteristic of R is 0.

(a) Let m be the least positive integer such that $m \cdot 1 = 0$, or let $m = 0$ if there is no such m. Show that m equals the characteristic of R.

(b) Show that the characteristic of an integral domain is either 0 or a prime number.

(c) Let R be a commutative ring and let $n \in \mathbf{Z}_{\geq 1}$. Show that the characteristic of $M_n(R)$ equals that of R.