Assignment 5 – All 2 parts – Math 412

Due: Thursday, Sept. 29, 2016, at the beginning of class

Textbook exercises:[1]

None yet.

Other exercises:

(1) Let $R = \mathbf{Q}$ be the field of rational numbers and let p be a prime number. Recall that every element r in $\mathbf{Q}$ can be written as $r = \frac{a}{b}$ with $a, b \in \mathbf{Z}$ and $\gcd(a, b) = 1$; we then say that $\frac{a}{b}$ is a reduced representation of r. Let

$$\mathbf{Z}_{(p)} := \left\{ \frac{a}{b} \in \mathbf{Q} : \frac{a}{b} \text{ is a reduced representation and } p \nmid b \right\},$$

i.e. the set of rational numbers whose denominator is not divisible by p.

(a) Show that $\mathbf{Z}_{(p)}$ is a subring of $\mathbf{Q}$.

(b) Show that if $n \in \mathbf{Z}$ is not divisible by p, then n is a unit in $\mathbf{Z}_{(p)}$.

(c) Show that p is not a unit in $\mathbf{Z}_{(p)}$.

(2) Let $S = \{a + b\sqrt[3]{2} + c\sqrt[3]{4} : a, b, c \in \mathbf{Z}\}$. Show that this is a subring of $\mathbf{R}$. It is denoted $\mathbf{Z}[\sqrt[3]{2}]$.

(3) (WI) Let R be a ring and let

$$Z(R) := \{z \in R : zr = rz \text{ for all } r \in R\}.$$

(a) Show that $Z(R) \leq R$. It is called the *centre of* R.

(b) Let R be the quaternions $\mathbf{H} = \{a + bi + cj + dk : a, b, c, d \in \mathbf{R}\}$ and let $S = \{a + bi \in \mathbf{H}\}$. Show that S is a commutative subring of $\mathbf{H}$, but there are elements in $\mathbf{H}$ that do not commute with elements of S.

(4) Let R be a commutative ring and let $n \in \mathbf{Z}_{\geq 1}$. Consider the ring $M_n(R)$ and its subset S consisting of the diagonal matrices whose diagonal entries are all equal,

[1] From Hungerford's *Abstract algebra, An introduction, Third edition*

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i.e.

$$S = \left\{ \begin{pmatrix} a & 0 & 0 & \dots & 0 \\ 0 & a & 0 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \dots & 0 & a & 0 \\ 0 & \dots & 0 & 0 & a \end{pmatrix} : a \in R \right\} \subseteq M_n(R).$$

(a) Show that S is a subring of the centre of $M_n(R)$.

(b) In fact, $S = Z(M_n(R))$. For simplicity, show this when $n = 2$. (Hint: start by supposing $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in the centre and compare $AE^{(1,1)}$ with $E^{(1,1)}A$; then compare $AE^{(1,2)}$ with $E^{(1,2)}A$, where

$$E^{(1,1)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad E^{(1,2)} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

(the general $n \times n$ case can be handled with generalizations of these matrices that one would denote $E^{(k,\ell)}$)).

(5) (WI) Let I be a nonempty index set and for each $i \in I$, let R_i be a nonzero ring. In class, we defined the ring $R = \prod_{i \in I} R_i$. Let

$$S := \{(r_i)_{i \in I} \in R : r_i = 0 \text{ for all but finitely many } i \in I\},$$

i.e. the set of all tuples in R with only finitely many nonzero components. Show that if I is infinite then S is *not* a subring of R.

(6) (WI) Let R be a ring that contains $\mathbf{Z}$ as a subring. Given an element $r \in R$, define, as in class, $\mathbf{Z}[r]$ to be the subring of R generated by $\mathbf{Z} \cup \{r\}$.

(a) Let $S := \{a_0 + a_1 r + a_2 r^2 + \cdots + a_n r^n : a_i \in \mathbf{Z} \text{ and } n \in \mathbf{Z}_{\geq 0}\}$. Show that $S \leq R$.

(b) Conclude that $\mathbf{Z}[r] = S$.

(c) Suppose r satisfies $c_0 + c_1 r + c_2 r^2 + \cdots + c_{d-1} r^{d-1} + r^d$ with $c_0, \dots, c_{d-1} \in \mathbf{Z}$. Show that $S = \{a_0 + a_1 r + a_2 r^2 + \cdots + a_{d-1} r^{d-1} : a_i \in \mathbf{Z}\}$.