Assignment 6 – All 2 parts – Math 412

Due: Thursday, Oct. 6, 2016, at the beginning of class

Textbook exercises:[1]

Section 3.3: 3, 12, 27(WI)

Other exercises:

(1) (WI) In class, we defined the polynomial ring $R[x]$ over a commutative ring R in a formal way as the set of tuples $(a_0, a_1, \dots)$ indexed by $\mathbf{Z}_{\geq 0}$ with only finitely many non-zero entries, with addition and multiplication defined by

$$(a_0, a_1, \dots) + (b_0, b_1, \dots) = (a_0 + b_0, a_1 + b_1, \dots)$$

and

$$(a_0, a_1, \dots) \cdot (b_0, b_1, \dots) = (c_0, c_1, \dots),$$

where

$$c_k = \sum_{i=0}^{k} a_i b_{k-i}.$$

(a) Show that the multiplication defined above is actually a binary operation on $R[x]$, i.e. that given two elements $(a_0, a_1, \dots)$ and $(b_0, b_1, \dots)$ in $R[x]$, their product $(a_0, a_1, \dots) \cdot (b_0, b_1, \dots)$ also lies in $R[x]$.

(b) Show that the element $(1, 0, 0, \dots) \in R[x]$ is an identity for the multiplication defined above.

(2) Let R be a ring and let $\mathcal{F}$ denote the ring $\mathrm{Func}(R, R)$ of R-valued functions on R. Given a polynomial

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n \in R[X]$$

with coefficients in R, we defined a function $F_f : R \to R$ by

$$F_f(r) = a_0 + a_1 r + \cdots + a_n r^n.$$

Let $\mathrm{Poly}(R, R) \subseteq \mathrm{Func}(R, R)$ be the functions $F : R \to R$ that are equal to F_f for some $f(x) \in R[x]$, i.e. $\mathrm{Poly}(R, R)$ is the set of functions $F : R \to R$ given by a

[1] From Hungerford's *Abstract algebra, An introduction, Third edition*

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formula

$$F(r) = a_0 + a_1 r + \cdots + a_n r^n,$$

with $a_i \in R$ and $n \in \mathbf{Z}_{\geq 0}$. We call $\mathrm{Poly}(R, R)$ the set of *polynomial functions* for R to itself.

(a) Show that $\mathrm{Poly}(R, R)$ is a subring of $\mathrm{Func}(R, R)$.

(b) For $R = \mathbf{Z}/2\mathbf{Z}$, show that every R-valued function on R is a polynomial function. (Hint: you can just find, by hand, a polynomial that gives every function.)

(c) Let S be a subring of R and let $\mathcal{F}_S \subseteq \mathrm{Func}(R, R)$ consist of the functions $f : R \to R$ whose image is inside of S, i.e. the set of functions $f : R \to R$ such that $f(r) \in S$ for all $r \in R$. Show that $\mathcal{F}_S$ is a subring of $\mathrm{Func}(R, R)$.

(3) Consider $R = \mathbf{Q}[x]$.

(a) Let S be the set of polynomials $f(x) = a_0 + a_1 x + a_2 x^2$ in $\mathbf{Q}[x]$, i.e. the polynomials whose coefficients of x^n is 0 for all $n \geq 3$. Is S a subring of R?

(b) Let S' be the set of polynomials $f(x) = a_0$ in $\mathbf{Q}[x]$, i.e. the polynomials whose coefficients of x^n is 0 for all $n \geq 1$. Is S' a subring of R?

(c) Let S'' be the set of polynomials $f(x) \in \mathbf{Q}[x]$ whose coefficients are in $\mathbf{Z}$. Is S'' a subring of R?

(4) (WI) Prove the following two statements made in class:

(a) Suppose $\varphi : R \to S$ is a ring homomorphism that is a bijection. Show that its inverse $\varphi^{-1} : S \to R$ is a homomorphism.

(b) Suppose $\varphi : R \to S$ is a bijection between two rings such that for all $r_1, r_2 \in R$

$$\varphi(r_1 + r_2) = \varphi(r_1) + \varphi(r_2) \quad \text{and} \quad \varphi(r_1 r_2) = \varphi(r_1)\varphi(r_2).$$

Show that $\varphi(1) = 1$.

(5) Recall the subring of $\mathbf{R}$ given by $\mathbf{Z}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbf{Z}\}$. Show that the function $a + b\sqrt{2} \mapsto a - b\sqrt{2}$ is an isomorphism from $\mathbf{Z}[\sqrt{2}]$ to itself. (An isomorphism from a ring to itself is called an *automorphism*.)