Assignment 7 – All 2 parts – Math 412

Due: Thursday, Oct. 13, 2016, at the beginning of class

Textbook exercises:[1]

Section 3.3: 31

Other exercises:

(1) Let $m, n \in \mathbf{Z}_{\geq 1}$.

(a) Suppose $n \mid m$. Define the function

$$\begin{array}{rcl} \varphi : \mathbf{Z}/m\mathbf{Z} & \to & \mathbf{Z}/n\mathbf{Z} \\ [a]_m & \mapsto & [a]_n \end{array}.$$

(where $[a]_m$ denotes the congruence class of a modulo m). Show that this function is well-defined and a ring homomorphism.

(b) Show that if $n \nmid m$, then there are no ring homomorphisms $\mathbf{Z}/m\mathbf{Z} \to \mathbf{Z}/n\mathbf{Z}$. (Hint: we did something about this in class.)

(2) (WI) Let $\varphi : R \to S$ be a ring homomorphism.

(a) Show that for all $r \in R$, $\varphi(-r) = -\varphi(r)$.

(b) Suppose $u \in R$ is a unit. Show that $\varphi(u)$ is a unit in S and $\varphi(u)^{-1} = \varphi(u^{-1})$.

(c) If $a \in R$ is a zerodivisor, is $\varphi(a)$ necessarily a zerodivisor in S? If so, prove it; if not, give a counterexample. (Hint: playing around with the homomorphisms in Question (1a) can help you out.)

(3) Let $m \in \mathbf{Z}_{\geq 2}$. Show that there are no ring homomorphisms $\mathbf{Q} \to \mathbf{Z}/m\mathbf{Z}$.

(4) Let $\varphi : R \to S$ and $\psi : S \to T$ be two ring homomophisms. Show that $\psi \circ \varphi : R \to T$ is a ring homomorphism.

(5) Suppose $R \cong S$. Show that their characteristics are equal.

[1]From Hungerford's *Abstract algebra, An introduction, Third edition* TURN OVER

(6) Find all solutions to the system of congruences

$$\begin{cases} x \equiv 1 \pmod 2 \\ x \equiv 1 \pmod 3 \\ x \equiv 5 \pmod 7. \end{cases}$$

(7) A Chinese Remainder problem where the moduli are not pairwise relatively prime.

(a) Consider the system of congruences

$$\begin{cases} x \equiv 2 \pmod 4 \\ x \equiv 1 \pmod{10}. \end{cases}$$

Does it have a solution? If so, find one. If not, explain why not.

(b) What about the system

$$\begin{cases} x \equiv 2 \pmod 4 \\ x \equiv 4 \pmod{10}? \end{cases}$$

(c) (WI) Suppose $m, n \in \mathbf{Z}_{\geq 2}$ and let $g = \gcd(m, n)$. Show that the system

$$\begin{cases} x \equiv a \pmod m \\ x \equiv b \pmod n \end{cases}$$

has a solution if and only if $a \equiv b \pmod g$.

(d) (WI) Suppose the system in the previous part has a solution $x_0 \in \mathbf{Z}$. Show that the set of solutions is $\{x_0 + Lk : k \in \mathbf{Z}\}$, where $L = \operatorname{lcm}(m, n)$.

(8) (WI) Suppose $I \trianglelefteq \mathbf{Z}$ is an ideal of $\mathbf{Z}$ that contains two non-zero integers $m, n \in \mathbf{Z}$. Show that I then contains their gcd.

(9) What is the kernel of the homomorphism in part (a) of question (1)?