Assignment 8 – Part 1 – Math 412

Textbook exercises:[1]

Section 4.1: 13

Other exercises:

(1) The *Euler totient function* is the function that sends a positive integer n to the number of units in $\mathbf{Z}/n\mathbf{Z}$, i.e.

$$\begin{aligned} \varphi : \mathbf{Z}_{\geq 1} &\to \mathbf{Z} \\ n &\mapsto \#\{u \in \mathbf{Z}/n\mathbf{Z} : u \text{ is a unit}\}. \end{aligned}$$

(a) Let R_1 and R_2 be rings and let $R = R_1 \times R_2$. Show that $a = (a_1, a_2) \in R$ is a unit if and only if a_i is a unit in R_i for $i = 1, 2$.

(b) Let m and n be relatively prime positive integers. Show that

$$\varphi(mn) = \varphi(m)\varphi(n).$$

(Hint: Chinese Remainder Theorem.)

(2) (WI) Let R be a commutative ring and let $f, g \in R[x]$.

(a) Show that $\deg(f + g) \leq \max(\deg(f), \deg(g))$.

(b) Show that if $\deg(f) \neq \deg(g)$, then $\deg(f + g) = \max(\deg(f), \deg(g))$.

(3) (a) Show that $1 + 2x$ is a unit in $(\mathbf{Z}/4\mathbf{Z})[x]$.

(b) Show that $1 + 10x$ is a unit in $(\mathbf{Z}/25\mathbf{Z})[x]$.

(c) Is $1 + 3x$ a unit in $(\mathbf{Z}/6\mathbf{Z})[x]$?

[1] From Hungerford's *Abstract algebra, An introduction, Third edition*