Assignment 10 – Part 1 – Math 412

Textbook exercises:[1]

Section 7.3: 27

Section 7.4: 3, 16

Other exercises:

(1) Let $X \neq \emptyset$ and let $Y \subseteq X$ be non-empty.

(a) Let $H := \{f \in S_X : f(Y) = Y\}$. Show that H is a subgroup of S_X. (Note: $f(Y) = Y$ means the image of the set Y under F is all of Y.)

(b) For $f \in S_X$, define $\text{res}(f) := f|_Y$. Show that res is a group homomorphism from H to S_Y.

(2) (a) In class, we defined a function $\varphi : G \to H$ between two groups to be a homomorphism if $\varphi(e) = e$, $\varphi(g_1 g_2) = \varphi(g_1)\varphi(g_2)$, and $\varphi(g^{-1}) = \varphi(g)^{-1}$, for all $g, g_1, g_2 \in G$. We then said that φ is a homomorphism if and only if $\varphi(g_1 g_2) = \varphi(g_1)\varphi(g_2)$ for all $g_1, g_2 \in G$. Prove this.

(b) In class, we said a group homomorphism $\varphi : G \to H$ was an isomorphism if there is a group homomorphism $\psi : H \to G$ such that $\varphi \circ \psi = \text{id}_H$ and $\psi \circ \varphi = \text{id}_G$. We then said that φ is an isomorphism if and only if φ is a bijection. Prove this.

(3) (WI) Let G and H be groups and let $\varphi : G \to H$ be a group homomorphism.

(a) Let $I \neq \emptyset$ be an index set and let $H_i \leq G$ for every $i \in I$. Show that $\bigcap_{i \in I} H_i$ is a subgroup of G.

(b) Show that $\text{im}\,\varphi$ is a subgroup of H.

(c) Let $H' \leq H$. Show that $\varphi^{-1}(H) \leq G$.

(4) (WI) Let G be a group.

(a) Let $g \in G$. The *centralizer of* g is $C(g) := \{h \in G : hg = gh\}$. Show that $C(g)$ is a subgroup of G.

(b) Let $S \subseteq G$. The *centralizer of* S is $C(S) := \{h \in G : hg = gh \text{ for all } g \in S\}$. Show that $C(S) = \bigcap_{g \in S} C(g)$ and that it is a subgroup of G.

(c) Show that $Z(G) = \bigcap_{g \in G} C(g)$.

[1] From Hungerford's *Abstract algebra, An introduction, Third edition*