**Assignment 10 – All 2 parts – Math 412**

**Due in class: Thursday, Nov. 21, 2019**

Textbook exercises:[1]

**Section 7.3:** 27

**Section 7.4:** 3, 16, 20(WI)

Other exercises:

(1) Let $X \neq \emptyset$ and let $Y \subseteq X$ be non-empty.

(a) Let $H := \{f \in S_X : f(Y) = Y\}$. Show that $H$ is a subgroup of $S_X$. (Note: $f(Y) = Y$ means the image of the set $Y$ under $F$ is all of $Y$.)

(b) For $f \in S_X$, define $\text{res}(f) := f|_Y$. Show that res is a group homomorphism from $H$ to $S_Y$.

(2) (a) In class, we defined a function $\varphi : G \to H$ between two groups to be a homomorphism if $\varphi(e) = e$, $\varphi(g_1g_2) = \varphi(g_1)\varphi(g_2)$, and $\varphi(g^{-1}) = \varphi(g)^{-1}$, for all $g, g_1, g_2 \in G$. We then said that $\varphi$ is a homomorphism if and only if $\varphi(g_1g_2) = \varphi(g_1)\varphi(g_2)$ for all $g_1, g_2 \in G$. Prove this.

(b) In class, we said a group homomorphism $\varphi : G \to H$ was an isomorphism if there is a group homomorphism $\psi : H \to G$ such that $\varphi \circ \psi = \text{id}_H$ and $\psi \circ \varphi = \text{id}_G$. We then said that $\varphi$ is an isomorphism if and only if $\varphi$ is a bijection. Prove this.

(3) (WI) Let $G$ and $H$ be groups and let $\varphi : G \to H$ be a group homomorphism.

(a) Let $I \neq \emptyset$ be an index set and let $H_i \leq G$ for every $i \in I$. Show that $\bigcap_{i \in I} H_i$ is a subgroup of $G$.

(b) Show that $\text{im}\,\varphi$ is a subgroup of $H$.

(c) Let $H' \leq H$. Show that $\varphi^{-1}(H) \leq G$.

(4) (WI) Let $G$ be a group.

(a) Let $g \in G$. The *centralizer of* $g$ is $C(g) := \{h \in G : hg = gh\}$. Show that $C(g)$ is a subgroup of $G$.

[1] From Hungerford's *Abstract algebra, An introduction, Third edition*

(b) Let $S \subseteq G$. The *centralizer of* $S$ is $C(S) := \{h \in G : hg = gh \text{ for all } g \in S\}$. Show that $C(S) = \bigcap_{g \in S} C(g)$ and that it is a subgroup of $G$.

(c) Show that $Z(G) = \bigcap_{g \in G} C(g)$.

(5) Let $G$ be a group with identity $e$.

(a) Suppose $g \neq e$. Why can't $g^2 = g$?

(b) Suppose $G = \{e, g, h\}$, i.e. $\#g = 3$. Write out the multiplication table of $G$ (Hint: first, this question is saying that there is only one group of order 3 up to multiplication; second, recall that we saw that every element of $G$ occurs exactly once in every row (and in every column) of the multiplication table of $G$.)